SOME THEOREMS ON FOURIER COEFFICIENTS

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I. **Trigonometric polynomials with coefficients** ± 1. Consider the trigonometric polynomial

$$(1.1) \qquad P(e^{i\theta}) = \sum_{n=1}^{N} \epsilon_n e^{in\theta}$$

where $\epsilon_n = \pm 1$. If we set $\|P\|_\infty = \max_\theta |P(e^{i\theta})|$, the Parseval theorem shows that $\|P\|_\infty \geqq N^{1/2}$, and the following problem arises: *does there exist an absolute constant A with the property that for each N one can find* $\epsilon_1, \cdots, \epsilon_N$, *equal to* ± 1, *so that*

$$(1.2) \qquad \|P\|_\infty \leqq AN^{1/2},$$

where P is given by (1.1)?

If one allows the coefficients ϵ_n to be complex numbers of absolute value 1, an affirmative answer to the question is furnished by the partial sums of the series $\sum_1^\infty e^{in\log n}e^{in\theta}$; this example is due to Hardy and Littlewood [**4**, pp. 116–118]. A theorem of Salem and Zygmund [**2**, pp. 270, 278] shows, roughly speaking, that $(N \log N)^{1/2}$ is the "most probable" order of magnitude for $\|P\|_\infty$ if $\epsilon_n = \pm 1$.

During the summer of 1958, Salem drew my attention to the problem stated in the first paragraph. It turns out that an affirmative answer can be given by a construction which uses nothing more sophisticated than the parallelogram law

$$(1.3) \qquad |\alpha + \beta|^2 + |\alpha - \beta|^2 = 2|\alpha|^2 + 2|\beta|^2.$$

After I found this construction I learned that the problem had been solved earlier, by essentially the same method, in the 1951 Master's Thesis of H. S. Shapiro [**3**]. Since the result is needed in Part II of this paper, I am publishing the proof here, with Shapiro's consent. As in the Hardy-Littlewood example, the polynomials (1.1) may actually be taken as the partial sums of a fixed series $\sum_1^\infty \epsilon_n e^{in\theta}$:

THEOREM I. *There exists a sequence* $\{\epsilon_n\}$ $(n = 1, 2, 3, \cdots)$, *with* $\epsilon_n = 1$ *or* -1, *such that*

$$(1.4) \qquad \left|\sum_{n=1}^{N} \epsilon_n e^{in\theta}\right| < 5N^{1/2} \qquad (0 \leqq \theta < 2\pi; N = 1, 2, 3, \cdots).$$

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PROOF. Set $P_0(x)=Q_0(x)=x$, and define polynomials P_k and Q_k inductively by

$$(1.5)\qquad \begin{cases} P_{k+1}(x) = P_k(x) + x^{2^k}Q_k(x), \\ Q_{k+1}(x) = P_k(x) - x^{2^k}Q_k(x), \end{cases} \qquad (k = 0, 1, 2, \cdots).$$

Then $P_k(e^{i\theta})$ is of the form (1.1), with $N=2^k$, and P_k is a partial sum of P_{k+1}. Hence we can define a sequence $\{\epsilon_n\}$ by setting ϵ_n equal to the nth coefficient of P_k, where $2^k>n$; this sequence will be shown to have the desired properties.

For $|x|=1$, (1.3) and (1.5) imply

$$\begin{aligned} |P_{k+1}(x)|^2 + |Q_{k+1}(x)|^2 &= |P_k(x) + x^{2^k}Q_k(x)|^2 + |P_k(x) - x^{2^k}Q_k(x)|^2 \\ &= 2|P_k(x)|^2 + 2|Q_k(x)|^2, \end{aligned}$$

and since $|P_0(x)|^2+|Q_0(x)|^2=2$, we conclude that

$$(1.6)\qquad |P_k(e^{i\theta})|^2 + |Q_k(e^{i\theta})|^2 = 2^{k+1}.$$

Hence

$$(1.7)\qquad |P_k(e^{i\theta})| \leqq 2^{1/2}\cdot 2^{k/2},$$

which proves (1.4) for $N=2^k$.

If now $s_n(P_k)$ and $s_n(Q_k)$ denote the nth partial sums of P_k and Q_k respectively, where $1\leqq n\leqq 2^k$, then

$$(1.8)\qquad \left.\begin{array}{l} |s_n(P_k)(e^{i\theta})| \\ |s_n(Q_k)(e^{i\theta})| \end{array}\right\} \leqq (2 + 2^{1/2})2^{k/2} \qquad (k = 0, 1, 2, \cdots).$$

This is obviously true if $k=0$. Suppose (1.8) holds for some k, and consider $s_n(P_{k+1})$ and $s_n(Q_{k+1})$, with $1\leqq n\leqq 2^{k+1}$. If $n\leqq 2^k$, (1.5) shows that

$$|s_n(P_{k+1})| = |s_n(Q_{k+1})| = |s_n(P_k)| < (2 + 2^{1/2})2^{(k+1)/2}.$$

If $2^k<n\leqq 2^{k+1}$, (1.5) and (1.7) show that

$$\begin{aligned} |s_n(P_{k+1})| &\leqq |P_k| + |\overset{k}{s}_{n-2}(Q_k)| \\ &\leqq 2^{(k+1)/2} + (2 + 2^{1/2})2^{k/2} = (2 + 2^{1/2})2^{(k+1)/2}. \end{aligned}$$

The same estimate holds for $|s_n(Q_{k+1})|$, and (1.8) is proved by induction.

To complete the proof of (1.4), suppose $2^{k-1}\leqq N\leqq 2^k$. By (1.8), we have

$$|s_N(P_k)(e^{i\theta})| \leqq (2 + 2^{1/2})2^{k/2} \leqq 2(1 + 2^{1/2})N^{1/2} < 5N^{1/2}.$$

II. **Transformations of Fourier coefficients.** In this section, p and

q will always denote conjugate exponents, i.e., $1/p+1/q=1$. For $1\leqq p<\infty$, L^p denotes the usual Lebesgue space of complex functions on the unit circle, normed by

$$\|f\|_p = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^p d\theta\right\}^{1/p}. \tag{2.1}$$

L^∞ is the space of all essentially bounded measurable functions on the circle. The Fourier coefficients of any $f\in L^1$ will be denoted by

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{i\theta})e^{-in\theta}d\theta \qquad (n = 0, \pm 1, \pm 2, \cdots). \tag{2.2}$$

If F is a complex function defined in the plane, we say that F maps A into B, where A and B are function spaces on the circle, if to every $f\in A$ there corresponds a $g\in B$ (we shall write: $g=F\circ f$) such that $\hat{g}=F(\hat{f})$. In other words, it is required that the series $\sum F(c_n)e^{in\theta}$ should be the Fourier series of a function in B whenever $\sum c_n e^{in\theta}$ is the Fourier series of a function in A.

The functions F which map L^1 into L^1 have recently been determined [1]; they are precisely those which are real-analytic near the origin (i.e., in some neighborhood of the origin); of course we must also have $F(0)=0$. For the other Lebesgue spaces, the situation is quite different. We first state some sufficient conditions:

THEOREM II. *Suppose $1<p\leqq 2$, and suppose there is a constant A such that $|F(z)|\leqq A|z|^{q/2}$ near the origin. Then F maps L^p into L^2.*

PROOF. If $f\in L^p$, the Hausdorff-Young theorem [4, p. 190] shows that $\sum|\hat{f}(n)|^q<\infty$, so that $\sum|F(\hat{f}(n))|^2<\infty$.

THEOREM III. *Suppose $1\leqq p\leqq 2$. If $|F(z)|\leqq A|z|^{2/p}$ near the origin, then F maps L^q into L^q.*

PROOF. If $f\in L^q$, then $\sum|\hat{f}(n)|^2<\infty$, so that $\sum|F(\hat{f}(n))|^p<\infty$, and the Hausdorff-Young theorem implies that $F\circ f\in L^q$.

REMARKS. 1. For $q=2$, this condition is necessary as well as sufficient.

2. For $q=\infty$, the hypothesis of Theorem III is: $|F(z)|\leqq A|z|^2$. It follows that F maps L^∞ (even L^2) into the class of functions which are sums of absolutely convergent trigonometric series.

3. If F is of the form

$$F(z) = a_1 z + a_2\bar{z} + |z|^{2/p}b(z), \tag{2.3}$$

where b is a function which is bounded near the origin, then F also maps L^q into L^q. Note that no smoothness conditions are imposed on

b (not even measurability is needed), in strong contrast to the results in [1].

I do not know whether (2.3) holds whenever F maps L^q into L^q. However, if we restrict ourselves to *even* functions F, Theorem I can be used to show that *Theorem* III *states a condition which is necessary as well as sufficient.* In fact, the following stronger assertion holds:

THEOREM IV. *Suppose* $1 \leqq p < \infty$, F *is an even function, and* $|\,|z|^{-2/p}F(z)\,|$ *is not bounded near the origin. Then there is a continuous function* f *on the circle to which corresponds no* $g \in L^q$ *with* $\hat{g} = F(\hat{f})$.

In other words, F does not map the space of all continuous functions into L^q, hence it does not map L^q into L^q.

PROOF. The hypothesis implies the existence of numbers $z_m \neq 0$ $(m = 1, 2, 3, \cdots)$, such that $m^2 z_m \to 0$ and $|F(z_m)| > m^5 |z_m|^{2/p}$. Define $N_m = [m^{-4} z_m^{-2}]$. These choices produce the relations

$$\sum_{m=1}^{\infty} |z_m| N_m^{1/2} < \infty \tag{2.4}$$

and

$$|F(z_m)| N_m^{1/p} \to \infty \text{ as } m \to \infty. \tag{2.5}$$

Now choose integers n_m so that

$$n_m + N_m < n_{m+1} - N_{m+1} \tag{2.6}$$

and define

$$T_m(e^{i\theta}) = z_m e^{in_m\theta}(\epsilon_1 e^{i\theta} + \cdots + \epsilon_{N_m} e^{iN_m\theta}), \tag{2.7}$$

where $\{\epsilon_n\}$ is the sequence of Theorem I. The series

$$f(e^{i\theta}) = \sum_{m=1}^{\infty} T_m(e^{i\theta}) \tag{2.8}$$

converges uniformly, by (2.4) and Theorem I, so that f is continuous.

Define the kernels K_m by

$$K_m(e^{i\theta}) = e^{i(n_m+N_m)\theta} \sum_{n=-2N_m}^{2N_m} \min\left(1, 2 - \frac{|n|}{N_m}\right) e^{in\theta}. \tag{2.9}$$

Suppose there is a function $g \in L^q$ such that $\hat{g} = F(\hat{f})$, i.e., $g = F \circ f$. Our choice of $\{n_m\}$ implies that $g * K_m = F \circ T_m$, where

$$(g * K_m)(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i(\theta-\phi)}) K_m(e^{i\phi}) d\phi. \tag{2.10}$$

Since $\|K_m\|_1 < 3$, we see that

$$\|F \circ T_m\|_q < 3\|g\|_q \qquad (m = 1, 2, 3, \cdots). \tag{2.11}$$

On the other hand, the assumption that $F(-z_m) = F(z_m)$ shows that

$$(F \circ T_m)(e^{i\theta}) = F(z_m)e^{inm\theta}(e^{i\theta} + \cdots + e^{iN_m\theta}), \tag{2.12}$$

so that

$$\left| (F \circ T_m)(e^{i\theta}) \right| = \left| F(z_m) \right| \cdot \left| \frac{\sin (N_m\theta/2)}{\sin (\theta/2)} \right| . \tag{2.13}$$

An easy computation now yields

$$\|F \circ T_m\|_q > C_q \left| F(z_m) \right| N_m^{1/p}, \tag{2.14}$$

where C_q is a positive constant, depending only on q. By (2.5), (2.14) implies that $\|F \circ T_m\|_q \to \infty$ as $m \to \infty$, and this contradicts (2.11).

The theorem follows.

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