Waring's Problem and the Hardy–Littlewood (Circle) Method

Sean Prendiville

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Abstract

The purpose of these notes is for me to test my understanding of the circle method as used in Waring's problem by giving an exposition. I have avoided proving the positivity of the singular integral and singular series as these were not techniques I needed at the time; although it could be argued that proving these facts actually constitutes the circle method, whilst the rest is merely error management

1 Notation

We will make constant use of Landau and Vinogradov asymptotic notation. Let f be a complex valued function defined on all large reals/integers and let g be a non-negative function with the same domain. Then it is usual to say that

$$f \ll g \qquad \text{or} \qquad f = O(g)$$

if there exists a constant $C > 0$ such that for all x

$$|f(x)| \leq Cg(x). \tag{1.1}$$

In what follows we will relax the above definition and require that (1.1) only holds for all large x. To state this precisely, we say $f \ll g$ or $f = O(g)$ if and only if there exists $N, C > 0$ such that for all $x \geq N$ we have $|f(x)| \leq Cg(x)$. It will often be the case that the implicit constant C and the cutoff value N depend on some parameters, say s, k, δ, other than x (of course C, N must be independent of x). In such a situation we write $f \ll_{s,k,\delta} g$ or $f = O_{s,k,\delta}(g)$ to indicate the dependence (when important). When $g = g_\varepsilon$ ($\varepsilon > 0$) is a family of functions indexed by ε, we will say $f \ll g_\varepsilon$ or $f = O(g_\varepsilon)$ if for every $\varepsilon > 0$ there exist $N_\varepsilon, C_\varepsilon > 0$ such that for all $x \geq N_\varepsilon$

$$|f(x)| \leq C_\varepsilon g_\varepsilon(x).$$

For example, $\log n \ll n^\varepsilon$.

2 Waring's Problem

In the eighteenth century the English mathematician Edward Waring commented that every natural number is a sum of at most 9 positive integral cubes and a sum of at most 19 biquadrates (fourth powers), and so on. No proof followed this assertion, but within a few years Lagrange had established the simplest result of this type, that every natural number is a sum of 4 squares (if we include zero as a square). In modern parlance, we interpret Waring's comment to be the conjecture that the cubes, fourth powers, fifth powers ('and so on') each form a *finite order basis* for the natural numbers. A subset $B \subset \mathbb{N}$ is called an (additive) basis if every natural number is equal to a sum of elements from B. The basis is of finite order s if we can ensure each of these sums has at most s summands. Lagrange's theorem therefore asserts that the squares form a basis of order 4. It wasn't until the turn of the 20th century that Hilbert managed to prove that for each k, the set of kth powers form a basis of finite order. The next obvious question is then: what is the smallest possible order $s = g(k)$ of the kth powers as an additive basis? It turns out that the value of $g(k)$ is all but dictated by some small integers which are particularly hard to represent efficiently as sums of kth powers, for instance

$$2^k \left\lfloor \left(\frac{3}{2}\right)^k \right\rfloor - 1.$$

A more interesting question is therefore to ask: what's the smallest possible value of s when s is any positive integer for which all but finitely many natural numbers can be represented as a sum of at most s kth powers. Let $G(k)$ denote the smallest possible such s. By considering the representation of the congruence class $7 \mod 8$ as a sum of three squares, we see that $G(2) = 4$. We will therefore assume $k \geq 3$ from now on.

In effect we are asking for the value of the smallest possible order of the kth powers as an *asymptotic basis*. A set $B \subset \mathbb{N}$ forms an asymptotic basis if all but finitely many natural numbers can be represented as a sum of elements from B. If each of these sums requires at most s terms, then we say B is an asymptotic basis of finite order s. Notice that when $1 \in B$ (as in the case of kth powers) then we have

$$B \text{ is an asymptotic basis of finite order} \quad \Longrightarrow \quad B \text{ is a basis of finite order.}$$

Let $R_s(n)$ denote the number of representations of the positive integer n as a sum of exactly s positive kth powers. To remove any ambiguity about the order of the summands, we mean

$$R_s(n) = \#\left\{(x_1, \ldots, x_s) \in \mathbb{N}^s : n = x_1^k + \cdots + x_s^k\right\}.$$

It's clear that $G(k) \leq s$ if $R_s(n) > 0$ for all large n. The aim of these notes is to establish this latter condition for $s = 2^k + 1$. It is a curious fact that often in combinatorial / geometric / number theoretic questions, the best bounds are obtained by transforming the question to one in analysis and answering it in this instance.[1] This is exactly what we'll do in this setting. How do we transform $R_s(n)$ into an analytic expression? The answer is via truncated generating functions and orthogonality relations. For $\alpha \in \mathbb{T}$ define $f(\alpha)$ to be the truncated generating function

$$\sum_{1 \leq x \leq X} e\left(\alpha x^k\right),$$

where $X = n^{1/k}$ and, as is usual in number theory, $e(\beta) := \exp(2\pi i \beta)$. Notice that

$$f(\alpha)^s = \sum_y R_s(y; X) e(\alpha y),$$

where $R_s(y; X)$ is equal to

$$\#\left\{(x_1, \ldots, x_s) \in ([1, X] \cap \mathbb{Z})^s : y = x_1^k + \cdots + x_s^k\right\}.$$

Happily $R_s(n) = R_s(n; n^{1/k})$. Using the orthogonality relation

$$\int_{\mathbb{T}} e(\alpha y) d\alpha = \begin{cases} 1 & \text{if } y = 0, \\ 0 & \text{if } y \in \mathbb{Z} \setminus \{0\}. \end{cases}$$

we have that

$$R_s(n) = \int_{\mathbb{T}} f(\alpha)^s e(-n\alpha) d\alpha,$$

which is our required analytic expression. Hardy and Littlewood noticed that when α is in a set $\mathfrak{M} \subset \mathbb{T}$ called the *major arcs*, consisting of those elements 'close' to a rational a/q with q 'small' (both 'close' and 'small' will be precisely defined later), then

$$f(\alpha) \approx f(a/q) \approx \frac{X}{q} \sum_{r=1}^{q} e\left(ar^k/q\right),$$

[1] Even more curiously, exact answers often seem to result only from pure algebraic/geometric arguments. The moral seems to be that for good approximations, forget the geometry and resort to analysis; whilst an exact answer requires the converse.

and the latter expression can be analysed using basic tools from analysis and multiplicative number theory, to establish that when s is large enough we have an asymptotic relation

$$\int_{\mathfrak{M}} f(\alpha)^s e(-n\alpha) d\alpha \sim C_{s,k}(n) n^{s/k-1} \tag{2.1}$$

for some non-negative function of n, $C_{s,k}(n)$, satisfying

$$1 \ll_{s,k} C_{s,k}(n) \ll_{s,k} 1.$$

Perhaps a more profound observation of Hardy and Littlewood is that when α lies in the *minor arcs* $\mathfrak{m} := \mathbb{T} \setminus \mathfrak{M}$ then there is sufficient cancellation in the exponential sum $f(\alpha)$ to ensure that when s is large enough

$$\int_{\mathfrak{m}} |f(\alpha)|^s \, d\alpha \ll n^{s/k-1-\tau}, \tag{2.2}$$

for some positive τ. Cancellation in an exponential sum is in some sense equivalent to the exponents being very uniformly spread in the circle $\mathbb{T}$. It is therefore not so surprising that Hardy and Littlewood's estimation for (2.2) is based on the seminal work of Weyl in 1916 on uniform distribution mod 1. Putting (2.1) and (2.2) together we have that for s sufficiently large

$$R_s(n) \sim C_{s,k}(n) n^{s/k-1}.$$

In particular, $R_s(n) > 0$ for all large enough n.

3 The Minor Arc Estimate

In estimating the integral over the minor arcs it turns out that we can obtain a very efficient mean value estimate of Hua when $s = 2^k$ (Hua's Lemma). To stretch this estimate to general $s > 2^k$ we will use the crude inequality

$$|f(\alpha)|^s \leq |f(\alpha)|^{2^k} \sup_{\alpha \in \mathfrak{m}} |f(\alpha)|^{s-k}.$$

We are therefore lead to bounding $|f(\alpha)|$. When $k = 1$, in which case the exponents in our exponential sum are linear, this is easy.

Lemma 3.1 (Linear Estimate)**.** *Let I be a subinterval of $[1, X]$. Then for all $\alpha \in \mathbb{T}$*

$$\sum_{x \in I} e(\alpha x) \leq \min\left\{X, \frac{1}{2}\|\alpha\|^{-1}\right\},$$

where

$$\min\left\{X, \frac{1}{2}0^{-1}\right\} := X$$

and

$$\|\alpha\| := \min\{|\beta| : \beta \equiv \alpha \mod 1\}.$$

Proof. The result is clear when $\alpha \in \mathbb{Z}$. Therefore suppose $\alpha \notin \mathbb{Z}$. Summing the geometric progression we have

$$\begin{aligned}\sum_{x \in I} e(\alpha x) &\leq \frac{2}{|e(\alpha) - 1|} \\ &= \frac{2}{|e(\alpha/2) - e(-\alpha/2)|} \\ &= \frac{1}{|\sin(\pi\alpha)|} \\ &= \frac{1}{\sin(\pi\|\alpha\|)}.\end{aligned}$$

Since $\sin(\pi x)$ is concave on $[0, 1/2]$ we have

$$\sin(\pi\|\alpha\|) \geq 2\|\alpha\|,$$

which suffices to establish the lemma. □

There is a method (Weyl differencing) for re-writing our sum $f(\alpha)$ as a sum of linear exponential sums. Applying the above lemma it follows that we eventually wish to estimate sums of the form

$$\sum_{x\le X} \min\left\{X, \|\alpha x\|^{-1}\right\}.$$

The next lemma which, along with its proof, appears somewhat complicated, allows us to bound sums of the above type.

Lemma 3.2 (Separation Lemma)**.** *Let $P,Q\ge 1$ be reals, $\alpha\in\mathbb{T}$, $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ $(=\mathbb{N}^+)$ with $(a,q)=1$ and $|\alpha-a/q|\le q^{-2}$. Then*

$$\sum_{x\le P} \min\left\{\frac{PQ}{x}, \|\alpha x\|^{-1}\right\} \ll PQ\left(q^{-1}+Q^{-1}+q(PQ)^{-1}\right)\log(2qP). \tag{3.1}$$

Moreover the implicit constant above can be taken to equal 42 (the answer to life, the universe and everything).

Proof. We break the sum S on the left hand side of (3.1) into intervals of length q to obtain

$$S\le \sum_{0\le j\le \frac{P}{q}} \sum_{r=0}^{q} \min\left\{\frac{PQ}{jq+r}, \|\alpha\left(jq+r\right)\|^{-1}\right\}.$$

By one of our assumptions there exists $|\theta|\le 1$ with

$$\alpha(jq+r) = \frac{\lfloor \alpha j q^2\rfloor + ar}{q} + \frac{\left\{\alpha j q^2\right\}}{q} + \frac{\theta r}{q^2}.$$

There are two cases to consider. (i) $j=0$ and $r\le q/2$; (ii) $j\ge 1$ or ($j=0$ and $r>q/2$). In case (i) for all r

$$\left\|\frac{\lfloor \alpha j q^2\rfloor + ar}{q} + \frac{\left\{\alpha j q^2\right\}}{q} + \frac{\theta r}{q^2}\right\| \ge \frac{1}{2}\left\|\frac{\lfloor \alpha j q^2\rfloor + ar}{q}\right\|. \tag{3.2}$$

In case (ii), (3.2) also holds, unless

$$\left\lfloor \alpha j q^2\right\rfloor + ar \equiv 0, \pm 1, \pm 2, \pm 3 \mod q. \tag{3.3}$$

Since $(a,q)=1$, (3.3) holds for at most 7 values of r (for each j); moreover, when it does hold in case (ii) we have

$$\frac{1}{jq+r}\le 2\frac{1}{(j+1)q}.$$

Putting all this together, S is at most

$$\begin{aligned}
&2\sum_{r\le q/2}\left\|\frac{ar}{q}\right\|^{-1} + 2\sum_{\substack{q/2<r\le q\\ q\nmid ar}}\left\|\frac{ar}{q}\right\|^{-1} + 14\frac{PQ}{q}\\
&\quad+\sum_{1\le j\le P/q}\left(2\sum_{\substack{1\le r\le q\\ q\nmid \lfloor \alpha j q^2\rfloor + ar}}\left\|\frac{\lfloor \alpha j q^2\rfloor + ar}{q}\right\|^{-1} + 14\frac{PQ}{(j+1)q}\right)\\
&\quad= 2\sum_{0\le j\le P/q}\sum_{\substack{1\le r\le q\\ q\nmid \lfloor \alpha j q^2\rfloor + ar}}\left\|\frac{\lfloor \alpha j q^2\rfloor + ar}{q}\right\|^{-1} + 14\sum_{0\le j\le P/q}\frac{PQ}{(j+1)q}\\
&\quad\le 4\sum_{0\le j\le P/q}\sum_{1\le r\le q/2}\frac{q}{r} + 14\sum_{0\le j\le P/q}\frac{PQ}{(j+1)q}\\
&\quad= 4q\left(\lfloor P/q\rfloor + 1\right)\sum_{1\le r\le q/2}\frac{1}{r} + 14\frac{PQ}{q}\sum_{0\le j\le P}\frac{1}{(j+1)}
\end{aligned}$$

For all real $T_1 \geq 1$ and $T_2 \geq 2$

$$\sum_{n \leq T_1} \frac{1}{n} \leq 2\log(2T_1) \qquad \sum_{n \leq T_2} \frac{1}{n} \leq 3\log(T_2).$$

Therefore

$$\begin{aligned} S &\leq 8q(P/q+1)\log(2q) + 42\frac{PQ}{q}\log(P+1) \\ &\leq 42\left(P + q + PQq^{-1}\right)\log(2qP) \end{aligned}$$

as required. □

3.1 Weyl Differencing

Let $\phi : G_1 \to G_2$ be a map between Abelian groups. We define the forward difference operator Δ inductively as follows

$$\Delta_{h_1}\phi(x) := \phi(x+h) - \phi(x);$$

whilst if $\psi = \Delta_{h_1,\ldots,h_n}\phi$ we set

$$\Delta_{h_1,\ldots,h_{n+1}}\phi(x) := \Delta_{h_{n+1}}\psi(x).$$

It can be checked that if σ is a permutation of $\{1, 2, \ldots, n\}$ then

$$\Delta_{h_{\sigma(1)},\ldots,h_{\sigma(n)}}\phi(x) = \Delta_{h_1,\ldots,h_n}\phi(x).$$

This is a simple consequence of the explicit formula

$$\Delta_{h_1,\ldots,h_n}\phi(x) = \sum_{\epsilon_1,\ldots,\epsilon_n \in \{0,1\}} (-1)^{\epsilon_1+\cdots+\epsilon_n-n}\phi(x + \epsilon_1 h_1 + \cdots + \epsilon_n h_n).$$

The reader is invited to check that when $p(x)$ is a polynomial over $R = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ of degree k with leading coefficient α_k, then for each $0 \leq j \leq k$ and $h_1, \ldots, h_j \in R$, $\Delta_{h_1,\ldots,h_j}p(x) = h_1 \cdots h_j q_j(x; \mathbf{h})$, where $q_j(x; \mathbf{h})$ is a polynomial over R of degree $k - j$ and whose coefficient of x^{k-j} equals $\frac{k!\alpha_k}{(k-j)!}$. We will use this fact later.

Lemma 3.3 (Weyl differencing lemma)**.** *Let $\phi : \mathbb{N} \to \mathbb{R}$ be a function. Define*

$$S_\phi(X) := \sum_{1 \leq x \leq X} e\left(\phi(x)\right).$$

Then

$$|S_\phi(X)|^{2^t} \leq (2X)^{2^t-t-1} \sum_{|h_1|<X} \cdots \sum_{|h_t|<X} \sum_{x \in I(h_1,\ldots,h_t)} e\left(\Delta_{h_1,\ldots,h_t}\phi(x)\right). \tag{3.4}$$

Where each $I(h_1, \ldots, h_t)$ is a sub-interval (possibly empty) of $[1, X]$ and $I(h_1, \ldots, h_{t+1}) \subset I(h_1, \ldots, h_t)$.

Proof. We'll proceed by induction on $t \geq 1$. Let I be a sub-interval of $[1, X]$. Then for any function $\psi : \mathbb{N} \to \mathbb{R}$

$$\begin{aligned} \left|\sum_{x \in I} e\left(\psi(x)\right)\right|^2 &= \sum_{x \in I}\sum_{y \in I} e\left(\psi(y) - \psi(x)\right) \\ &= \sum_{x \in I}\sum_{h \in I-x} e\left(\psi(x+h) - \psi(x)\right) \\ &= \sum_{|h|<X}\sum_{x \in I \cap (I-h)} e\left(\Delta_h\psi(x)\right). \end{aligned}$$

Let us call this identity the 'Weyl differencing trick'. In particular if $I(h_1) := [1, X] \cap ([1, X] - h_1)$ then

$$|S_\phi(X)|^2 = \sum_{|h_1|<X}\sum_{x \in I(h_1)} e\left(\Delta_{h_1}\phi(x)\right),$$

which gives us (3.4) for $t=1$. Next suppose the result holds for a specific $t \geq 1$. Combining this with the Cauchy-Schwarz inequality, followed by another application of the Weyl differencing trick we have

$$\begin{aligned}|S_\phi(X)|^{2^{t+1}} &\leq \left((2X)^{2^t-t-1}\sum_{|h_1|<X}\cdots\sum_{|h_t|<X}\sum_{x\in I(h_1,\ldots,h_t)} e\left(\Delta_{h_1,\ldots,h_t}\phi(x)\right)\right)^2 \\ &\leq (2X)^{2^{t+1}-2t-2}(2X)^t \sum_{|h_1|<X}\cdots\sum_{|h_t|<X}\left|\sum_{x\in I(h_1,\ldots,h_t)} e\left(\Delta_{h_1,\ldots,h_t}\phi(x)\right)\right|^2 \\ &= (2X)^{2^{t+1}-(t+1)-1}\sum_{|h_1|<X}\cdots\sum_{|h_t|<X}\sum_{|h_{t+1}|<X}\sum_{x\in I(h_1,\ldots,h_t,h_{t+1})} e\left(\Delta_{h_1,\ldots,h_t,h_{t+1}}\phi(x)\right),\end{aligned}$$

where $I(h_1,\ldots,h_t,h_{t+1}) := I(h_1,\ldots,h_t)\cap(I(h_1,\ldots,h_t)-h_{t+1})$. □

Taking ϕ to be a polynomial p of degree k over $\mathbb{R}$ with leading coefficient α_k and taking $t=k-1$ we have that for each choice of integers $h_1,\ldots,h_{k-1}$ in the interval $(-X,X)$,

$$\Delta_{h_1,\ldots,h_{k-1}}p(x) = k!h_1\cdots h_{k-1}\alpha_k x + \beta(h_1,\ldots,h_{k-1}),$$

for some constant $\beta(h_1,\ldots,h_{k-1})$ independent of x. Inputting this into the Weyl differencing lemma we have

$$\left|\sum_{1\leq x\leq X} e\left(p(x)\right)\right|^{2^{k-1}} \leq (2X)^{2^{k-1}-k}\sum_{|h_1|<X}\cdots\sum_{|h_{k-1}|<X}\left|\sum_{x\in I(h_1,\ldots,h_{k-1})} e\left(k!h_1\cdots h_{k-1}\alpha_k x\right)\right|. \tag{3.5}$$

By the linear estimate (Lemma 3.1)

$$\left|\sum_{x\in I(h_1,\ldots,h_{k-1})} e\left(k!h_1\cdots h_{k-1}\alpha_k x\right)\right| \leq \min\left\{X, \|k!h_1\cdots h_{k-1}\alpha_k\|^{-1}\right\}.$$

Collecting the terms in the sum (3.5) for which some $h_i=0$ and ignoring the sign of each non-zero h_i, we have

$$\left|\sum_{1\leq x\leq X} e\left(p(x)\right)\right|^{2^{k-1}} \leq (2X)^{2^{k-1}-k}\sum_{|h_1|<X}\cdots\sum_{|h_{k-1}|<X}\min\left\{X,\|k!h_1\cdots h_{k-1}\alpha_k\|^{-1}\right\} \tag{3.6}$$

$$\leq (k-1)(2X)^{2^{k-1}-1} + 2^{k-1}(2X)^{2^{k-1}-k}\sum_{1\leq h_1<X}\cdots\sum_{1\leq h_{k-1}<X}\min\left\{X,\|k!h_1\cdots h_{k-1}\alpha_k\|^{-1}\right\} \tag{3.7}$$

$$\leq (k-1)(2X)^{2^{k-1}-1} + 2^{k-1}(2X)^{2^{k-1}-k}\sum_{1\leq n<X^{k-1}} d_{k-1}(n)\min\left\{X,\|k!n\alpha_k\|^{-1}\right\}, \tag{3.8}$$

where $d_{k-1}(n) := \#\left\{(a_1,\ldots,a_{k-1})\in\mathbb{N}^{k-1} : a_1\cdots a_{k-1}=n\right\}$ is the $(k-1)$-fold iterated divisor function. It can be easily established that $d_{k-1}(n) \ll_{\varepsilon,k} n^\varepsilon$, see for example pages 5–6 of the lecture notes [Bro07]. It follows that

$$\left|\sum_{1\leq x\leq X} e\left(p(x)\right)\right|^{2^{k-1}} \ll_{\varepsilon,k} X^{2^{k-1}-1} + X^{2^{k-1}-k+\varepsilon}\sum_{1\leq n\leq k!X^{k-1}}\min\left\{X,\|n\alpha_k\|^{-1}\right\} \tag{3.9}$$

Let $a\in\mathbb{Z}$, $q\in\mathbb{N}$ satisfy $q\leq X^k$, $(a,q)=1$ and $|\alpha_k - a/q|\leq q^{-2}$. Then applying the separation lemma with $P=k!X^{k-1}, Q=X$ to the rightmost sum in (3.9) we have that for all $X\geq 1$

$$\left|\sum_{1\leq x\leq X} e\left(p(x)\right)\right|^{2^{k-1}} \ll_{\varepsilon,k} X^{2^{k-1}-1} + X^{2^{k-1}+\varepsilon}\left(q^{-1}+X^{-1}+qX^{-k}\right). \tag{3.10}$$

Consequently, we've obtained:

Lemma 3.4 (Weyl's Inequality)**.** *Let p be a polynomial of degree k over $\mathbb{R}$ with leading coefficient α_k. Suppose there are integers $a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a,q)=1$ and $|\alpha_k - a/q| \leq q^{-2}$. Then*

$$\left|\sum_{1\leq x\leq X} e\left(p(x)\right)\right|^{2^{k-1}} \ll_{\varepsilon,k} X^{2^{k-1}+\varepsilon}\left(q^{-1}+X^{-1}+qX^{-k}\right), \tag{3.11}$$

or more conveniently

$$\left|\sum_{1\leq x\leq X} e\left(p(x)\right)\right| \ll_{\varepsilon,k} X^{1+\varepsilon}\left(q^{-1}+X^{-1}+qX^{-k}\right)^{2^{1-k}}. \tag{3.12}$$

Proof. The discussion before the statement of the lemma establishes the inequality provided $q \leq X^k$. If $q > X^k$ then it is a triviality. □

Notice that if $|\alpha - a/q| \leq q^{-1}X^{1-k}$ where $(a,q)=1$ and $X^{\theta} < q \leq X^{k-\theta}$ for some $\theta \in (0,1]$, then Weyl's inequality gives us that

$$f(\alpha) := \sum_{1\leq x\leq X} e\left(\alpha x^k\right) \ll_{\varepsilon,k} X^{1-\theta 2^{1-k}+\varepsilon},$$

and so

$$f(\alpha)^s \ll X^{s-s\theta 2^{1-k}+\varepsilon},$$

which is a small saving from the trivial estimate. We can gain further savings by using a mean value estimate of Hua.

3.2 Hua's Lemma

Lemma 3.5 (Hua's Lemma)**.** *Let $f(\alpha) = \sum_{1\leq x\leq X} e\left(\alpha x^k\right)$. Then for each $1 \leq j \leq k$*

$$\int_{\mathbb{T}} |f(\alpha)|^{2^j}\, d\alpha \ll_{j,\varepsilon,k} X^{2^j-j+\varepsilon}. \tag{3.13}$$

Proof. We proceed by induction on $1 \leq j \leq k$. For $j=1$ we save an extra epsilon:

$$\int_{\mathbb{T}} |f(\alpha)|^2\, d\alpha = \#\left\{(x,y) \in [X]^2 : x^k = y^k\right\} = \lfloor X \rfloor.$$

Next suppose the result holds for $1 \leq j < k$. By the Weyl differencing lemma

$$|f(\alpha)|^{2^j} \leq (2X)^{2^j-j-1} \sum_{|h_1|<X} \cdots \sum_{|h_j|<X} \sum_{x\in I(\mathbf{h})} e\left(\alpha \Delta_{\mathbf{h}}(x^k)\right).$$

Therefore

$$\begin{aligned}
\int_{\mathbb{T}} |f(\alpha)|^{2^{j+1}}\, d\alpha &= \int_{\mathbb{T}} |f(\alpha)|^{2^j} |f(\alpha)|^{2^j}\, d\alpha && (3.14)\\
&\leq (2X)^{2^j-j-1} \sum_{|h_1|<X} \cdots \sum_{|h_j|<X} \sum_{x\in I(\mathbf{h})} \int_{\mathbb{T}} |f(\alpha)|^{2^j} e\left(\alpha\Delta_{\mathbf{h}}(x^k)\right) d\alpha && (3.15)\\
&= (2X)^{2^j-j-1} M, && (3.16)
\end{aligned}$$

where M denotes the number of tuples $(\mathbf{h}, \mathbf{x}, \mathbf{y}, x)$ with $\mathbf{h} \in (-X,X)^j$, $\mathbf{x}, \mathbf{y} \in [X]^{2^{j-1}}$ and $x \in I(\mathbf{h})$ for which

$$\sum_{i=1}^{2^{j-1}} \left(x_i^k - y_i^k\right) = \Delta_{\mathbf{h}}(x^k) = h_1 \cdots h_j q_j(x;\mathbf{h}), \tag{3.17}$$

where $q_j(x;\mathbf{h})$ is a polynomial of degree $k-j$ with leading coefficient equal to $k!/(k-j)!$ (see the remarks before the Weyl differencing lemma). In particular, $q_j(x;\mathbf{h})$ is a non-zero polynomial in x, regardless of the value of $\mathbf{h}$. Let

$$N(\mathbf{x},\mathbf{y}) := \sum_{i=1}^{2^{j-1}} \left(x_i^k - y_i^k\right).$$

First we'll count those tuples $(\mathbf{h},\mathbf{x},\mathbf{y},x)$ for which $N(\mathbf{x},\mathbf{y})=0$. By induction, there are $\ll_{j,\varepsilon,k} X^{2^j-j+\varepsilon}$ choices for $(\mathbf{x},\mathbf{y})$. $(\mathbf{h},x)$ must also satisfy $\Delta_{\mathbf{h}}\left(x^k\right)=0$, so either $h_i=0$ for some i or x is a root of $q_j(x;\mathbf{h})$. Hence in this case there are at most $jX^j+(k-j)X^j=kX^j$ choices for $(\mathbf{h},x)$. The tuples $(\mathbf{h},\mathbf{x},\mathbf{y},x)$ with $N(\mathbf{x},\mathbf{y})=0$ therefore contribute

$$\ll_{j,\varepsilon,k} X^{2^j+\varepsilon}.$$

Next we'll count those tuples $(\mathbf{h},\mathbf{x},\mathbf{y},x)$ with $N(\mathbf{x},\mathbf{y})\neq 0$. Trivially there are at most X^{2^j} choices for $(\mathbf{x},\mathbf{y})$. For each of these choices, we have

$$\left(|h_1|,\ldots,|h_j|,|q_j(x;\mathbf{h})|\right) \in \left\{(a_1,\ldots,a_{j+1})\in\mathbb{N}^{j+1} : a_1\cdots a_{j+1} = |N(x,y)|\right\}.$$

Where the latter set has size $d_{j+1}\left(|N(\mathbf{x},\mathbf{y})|\right) \ll_{\varepsilon,j} |N(\mathbf{x},\mathbf{y})|^{\varepsilon}$. Since $|N(\mathbf{x},\mathbf{y})|\le 2^{j-1}X^k$, the set has size

$$\ll_{\varepsilon,j,k} X^{\varepsilon}.$$

For each $(a_1,\ldots,a_{j+1})\in\mathbb{N}^{j+1}$, there are at most $2^{j+1}(k-j)$ tuples $(\mathbf{h},x)$ satisfying

$$\left(|h_1|,\ldots,|h_j|,|q_j(x;\mathbf{h})|\right) = (a_1,\ldots,a_{j+1}).$$

Hence the $(\mathbf{h},\mathbf{x},\mathbf{y},x)$ with $N(\mathbf{x},\mathbf{y})\neq 0$ contribute

$$\ll_{\varepsilon,j,k} X^{2^j+\varepsilon}.$$

Combining both cases with (3.16), we obtain the desired result. □

We now have all the tools at our disposal to obtain our minor arc estimate, however we've yet to even define our minor arcs (we hope to make them as small as possible). We will therefore have to wait and see the limits of our major arc technology before we can perform this estimate.

3.3 Alternate Derivation of Weyl's Inequality

I find the following derivation of the separation lemma (3.1) slightly less opaque than that given above. The treatment is based on that found in §3 of [Gow].

We start with a useful inequality which is the main ingredient for the separation lemma. We say the real numbers $\theta_1,\ldots,\theta_n$ are δ-separated if

$$\|\theta_i-\theta_j\| \ge \delta \quad \text{when } i\neq j. \tag{3.18}$$

Notice that if (3.18) holds then $\delta\le 1/2$.

Lemma 3.6 (Proto-Separation Lemma). *Let $P\ge 1$, $Q\ge 2$ and $\delta>0$. Suppose $P\delta\le 1$ and $(\theta_j)_{j\le P}$ are δ-separated. Then*

$$S := \sum_{j\le P} \min\left\{Q, \|\theta_j\|^{-1}\right\} \le 8\left(Q+\delta^{-1}\right)\log Q. \tag{3.19}$$

Proof. Clearly we may assume $\theta_j\in[-\frac{1}{2},\frac{1}{2}]$ for all j, and so $\|\theta_j\|=|\theta_j|$. Let S^+ denote the sum restricted to non-negative θ_j and S^- the sum restricted to negative θ_j. Then

$$S \le 2\max\left\{S^+,S^-\right\}.$$

Multiplying the sequence θ_j by -1 (if necessary), we may assume $\max\left\{S^+,S^-\right\}=S^+$. Re-ordering if necessary, we may assume

$$0\le\theta_1<\theta_2<\cdots<\theta_k$$

and $\theta_j < 0$ for all $j > k$. Set $t := 1 + \left\lceil \frac{1}{\delta Q} \right\rceil \geq 2$. For each $j \in [1, k]$

$$\begin{aligned}
\theta_j - \theta_{j-1} &= \theta_1 + \sum_{i=2}^{j} \theta_i - \theta_{i-1} \\
&= \theta_1 + \sum_{i=2}^{j} |\theta_i - \theta_{i-1}| \\
&\geq \theta_1 + \sum_{i=2}^{j} \|\theta_i - \theta_{i-1}\| \\
&\geq (j-1)\delta.
\end{aligned}$$

Hence if $j \in [t+1, k]$, then $\theta_j \geq \frac{1}{Q}$ and so

$$\sum_{j=t+1}^{k} \min\left\{Q, \|\theta_j\|^{-1}\right\} = \sum_{j=t+1}^{k} \theta_j^{-1} \leq \delta^{-1} \sum_{j=t+1}^{k} \frac{1}{j-1}.$$

The following inequalities can easily be checked

$$\log X \leq \sum_{1 \leq n \leq X} \frac{1}{n} \leq \log X + 1 \qquad (X \geq 1).$$

Therefore

$$\begin{aligned}
S^{+} &\leq tQ + \delta^{-1} \sum_{j=2}^{k} \frac{1}{j-1} - \delta^{-1} \sum_{j=2}^{t} \frac{1}{j-1} \\
&= tQ + \delta^{-1} \sum_{j=1}^{k-1} \frac{1}{j} - \delta^{-1} \sum_{j=1}^{t-1} \frac{1}{j} \\
&\leq \left(2Q + \delta^{-1}\right) + \delta^{-1} \left(\log(k-1) + 1 - \log(t-1)\right) \\
&\leq \left(2Q + \delta^{-1}\right) + \delta^{-1} \left(\log P + 1 - \log\left(\frac{1}{\delta}\right) + \log Q\right) \\
&\leq \left(2Q + 2\delta^{-1}\right) + \delta^{-1} \log Q.
\end{aligned}$$

The last inequality follows because $P \leq \delta^{-1}$. Since $Q \geq 2$, $\frac{3}{2} \log Q \geq 1$, and so

$$S^{+} \leq 3\left(Q + \delta^{-1}\right) \log Q + \left(Q + \delta^{-1}\right) \log Q.$$

The result now follows. □

We can now derive a slightly weaker version of the separation lemma:

Lemma 3.7 (Separation Lemma)**.** *Let $P \geq 1$, $Q \geq 2$ and let α be real with approximation $|\alpha - a/q| \leq q^{-2}$, where $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $(a, q) = 1$. Then for any $\beta \in \mathbb{R}$*

$$S := \sum_{x \leq P} \min\left\{Q, \|x\alpha + \beta\|^{-1}\right\} \leq 32\left(q^{-1} + P^{-1} + Q^{-1} + q\,(PQ)^{-1}\right) PQ \log Q. \tag{3.20}$$

Proof. When P is small enough, i.e. $P \leq \lfloor q/2 \rfloor + 1$, then we can do much better than (3.20). Let $n := \lfloor q/2 \rfloor + 1$ and $\theta_j := j\alpha + \beta$ $(j = 1, \ldots, n)$. Then for $1 \leq i < j \leq n$ we have $i \not\equiv j \mod q$ and hence

$$\begin{aligned}
\|\theta_j - \theta_i\| &= \left\| \frac{a}{q}(j-i) + \left(\alpha - \frac{a}{q}\right)(j-i) \right\| \\
&\geq \left\| \frac{a(j-i)}{q} \right\| - \left|\alpha - \frac{a}{q}\right| (j-i) \\
&\geq \frac{1}{q} - \frac{1}{2q} = \frac{1}{2q}.
\end{aligned}$$

Thus the θ_j are $(2q)^{-1}$-separated, also $n(2q)^{-1} \le \frac{1}{4} + (2q)^{-1} \le 1$. Hence by Lemma 3.6, for any $\beta \in \mathbb{R}$ and any $P' \le n$

$$\sum_{x \le P'} \min\left\{Q, \|x\alpha + \beta\|^{-1}\right\} \le 8(Q + 2q)\log Q \tag{3.21}$$

Notice that if $k \in \mathbb{N}$ we can take $\beta_k := kn\alpha + \beta$ to ensure

$$\sum_{x=kn+1}^{(k+1)n} \min\left\{Q, \|x\alpha + \beta\|^{-1}\right\} \le 8(Q + 2q)\log Q.$$

Therefore

$$\begin{aligned}
\sum_{x \le P} \min\left\{Q, \|x\alpha + \beta\|^{-1}\right\} &\le \sum_{0 \le k \le P/n} \sum_{x=kn+1}^{(k+1)n} \min\left\{Q, \|x\alpha + \beta\|^{-1}\right\} \\
&\le \left(\frac{P}{n} + 1\right) 8(Q + 2q)\log Q \\
&\le \left(\frac{2P}{q} + 1\right) 8(Q + 2q)\log Q \\
&= 8\left(\frac{2PQ}{q} + Q + 4P + 2q\right)\log Q \\
&\le 32\left(q^{-1} + P^{-1} + Q^{-1} + q\,(PQ)^{-1}\right) PQ\log Q.
\end{aligned}$$

□

There is one more ingredient for Weyl's inequality which we will give an alternative proof for.

Lemma 3.8 (Order of the Divisor Function)**.** *If $n \ge e^{e^4}$, then it has at most $n^{\frac{4}{\log\log n}}$ divisors.*

Proof. Let $n = \prod_{i=1}^{s} p_i^{a_i}$, where $p_1, \ldots, p_s$ are distinct primes and $a_1, \ldots, a_s$ are positive integers. Then for any t

$$\begin{aligned}
d(n) = \prod_{i=1}^{s}(a_i + 1) &= \prod_{\substack{i \\ p_i \le t}} (a_i + 1) \prod_{\substack{i \\ p_i > t}} (a_i + 1) \\
&\le \left(\frac{\log n}{\log 2} + 1\right)^t \prod_{\substack{i \\ p_i > t}} 2^{a_i} \\
&\le \left(\frac{\log n}{\log 2} + 1\right)^t \prod_{\substack{i \\ p_i > t}} p_i^{a_i \log 2/\log t} \\
&\le \exp\left(t\log\left(\frac{\log n}{\log 2} + 1\right)\right) n^{\frac{\log 2}{\log t}} \\
&= \exp\left(t\log\left(\frac{\log n}{\log 2} + 1\right) + \frac{\log n \log 2}{\log t}\right) \\
&\le \exp\left(t\log 3 + t\log\log n + \frac{\log n \log 2}{\log t}\right).
\end{aligned}$$

Differentiating the argument of the last exponential function and performing some rough approximations, we see that the last expression is minimised when t is about $\log n/(\log\log n)^2$. Setting $t := \log n/(\log\log n)^2$, we get

$$t\log 3 + t\log\log n + \frac{\log n \log 2}{\log t} = \frac{\log n \log 3}{(\log\log n)^2} + \frac{\log n}{\log\log n} + \frac{\log n \log 2}{\log\log n - 2\log\log\log n}.$$

It therefore suffices to show

$$\frac{\log 3}{(\log\log n)^2} + \frac{\log 2}{\log\log n - 2\log\log\log n} \le \frac{3}{\log\log n}.$$

Since $n \geq e^{e^4}$, $\log\log n \geq 4$, so we would like to show:

$$\frac{\log 2}{\log\log n - 2\log\log\log n} \leq \frac{3 - \frac{1}{4}\log 3}{\log\log n}.$$

We would be done if

$$2\left(3 - \frac{1}{4}\log 3\right)\log\log\log n \leq \left(3 - \frac{1}{4}\log 3 - \log 2\right)\log\log n. \tag{3.22}$$

Let

$$f(x) := \left(3 - \frac{1}{4}\log 3 - \log 2\right)x - \left(6 - \frac{1}{2}\log 3\right)\log x,$$

then

$$f'(x) = \left(3 - \frac{1}{4}\log 3 - \log 2\right) - \frac{6 - \frac{1}{2}\log 3}{x} \geq 0 \iff x \geq \frac{6 - \frac{1}{2}\log 3}{3 - \frac{1}{4}\log 3 - \log 2}.$$

In particular, f is increasing on $[4, \infty)$. (3.22) therefore follows if we can show

$$\left(6 - \frac{1}{2}\log 3\right)\log\log\log e^{e^4} \leq \left(3 - \frac{1}{4}\log 3 - \log 2\right)\log\log e^{e^4}.$$

This is equivalent to checking

$$(12 - \log 3)\log 2 \leq (12 - \log 3 - 4\log 2),$$

or

$$16\log 2 + (1 - \log 2)\log 3 \leq 12.$$
$$16\log 2 \leq 11.2,$$

hence it remains to show

$$(1 - \log 2)\log 3 \leq 0.8$$

This follows since $\log 2 \geq 1/2$ and $\log 3 \leq 1.5$. □

Corollary 3.9. *For all $n \geq e^{e^{4k/\varepsilon}}$*

$$d_k(n) \leq n^{\varepsilon}$$

and therefore for all $n \in \mathbb{N}$

$$d_k(n) \leq e^{(1-\varepsilon)e^{4k/\varepsilon}} n^{\varepsilon}.$$

Proof.

$$\frac{4}{\log\log n} \leq \varepsilon/k,$$

so

$$d_k(n) \leq d(n)^k \leq \left(n^{\varepsilon/k}\right)^k.$$

□

2nd Proof of Weyl's Inequality. We begin from (3.8):

$$\left|\sum_{1\leq x\leq X} e\left(p(x)\right)\right|^{2^{k-1}} \leq (k-1)(2X)^{2^{k-1}-1} + 2^{k-1}(2X)^{2^{k-1}-k} \sum_{1\leq n<X^{k-1}} d_{k-1}(n)\min\left\{X, \|k!n\alpha_k\|^{-1}\right\}.$$

It follows from Corollary 3.9 that for $1 \leq n < X^{k-1}$

$$d_{k-1}(n) \leq e^{e^{4k^2/\varepsilon}} n^{\varepsilon/(k-1)} \leq e^{e^{4k^2/\varepsilon}} X^{\varepsilon}.$$

Therefore

$$\left|\sum_{1\leq x\leq X} e\left(p(x)\right)\right|^{2^{k-1}} \leq (k-1)(2X)^{2^{k-1}-1} + e^{e^{4k^2/\varepsilon}} 2^{2^{k-1}-1} X^{2^{k-1}-k+\varepsilon} \sum_{1\leq n\leq k!X^{k-1}} \min\left\{X, \|k!n\alpha_k\|^{-1}\right\}. \tag{3.23}$$

Let $a \in \mathbb{Z}$, $q \in \mathbb{N}$ satisfy $q \le X^k$, $(a,q) = 1$ and $|\alpha_k - a/q| \le q^{-2}$. Then applying the second version of the separation lemma (Lemma 3.7 with $Q = X$ and $P = k!X^{k-1}$ to the rightmost sum in (3.23) we have that for all $X \ge 2$

$$\begin{aligned}\left|\sum_{1\le x\le X} e\left(p(x)\right)\right|^{2^{k-1}} &\le e^{e^{4k^2/\varepsilon}} 2^{2^{k-1}-1}\left(X^{2^{k-1}-1} + X^{2^{k-1}+\varepsilon} 32k!\left(\frac{1}{q} + \frac{1}{k!X^{k-1}} + \frac{1}{X} + \frac{q}{k!X^k}\right)\log X\right)\\ &\le e^{e^{4k^2/\varepsilon}} 2^{2^{k-1}-1} X^{2^{k-1}+\varepsilon} 32k!\left(\frac{1}{q} + \frac{1}{X} + \frac{q}{X^k}\right)\log X.\end{aligned}$$

It can be checked that $\log x \le x^{\varepsilon}$ for all $x \ge \varepsilon^{-\frac{1}{\varepsilon}}$. Therefore $\log X \le \varepsilon^{-\frac{1}{\varepsilon}} X^{\varepsilon}$. Hence

$$\left|\sum_{1\le x\le X} e\left(p(x)\right)\right|^{2^{k-1}} \le k!(\varepsilon/2)^{-\frac{1}{\varepsilon/2}} e^{e^{8k^2/\varepsilon}} 2^{2^{k-1}+4} X^{2^{k-1}+\varepsilon}\left(\frac{1}{q} + \frac{1}{X} + \frac{q}{X^k}\right).$$

Consequently, we've obtained:

Lemma 3.10 (Weyl's Inequality with Constant)**.** *Let p be a polynomial of degree k over $\mathbb{R}$ with leading coefficient α_k. Suppose there are integers $a \in \mathbb{Z}$, $q \in \mathbb{N}$ with $(a,q) = 1$ and $|\alpha_k - a/q| \le q^{-2}$. Then*

$$\left|\sum_{1\le x\le X} e\left(p(x)\right)\right|^{2^{k-1}} \le C_{\varepsilon,k} X^{2^{k-1}+\varepsilon}\left(q^{-1} + X^{-1} + qX^{-k}\right), \tag{3.24}$$

where

$$C_{\varepsilon,k} = k!(\varepsilon/2)^{-\frac{1}{\varepsilon/2}} e^{e^{8k^2/\varepsilon}} 2^{2^{k-1}+4}. \tag{3.25}$$

□

4 The Major Arcs

4.1 An Asymptotic for $f(\alpha)$

Let $a \in \mathbb{Z}$, $q \in \mathbb{N}$ and $\beta = \alpha - a/q$. Dividing our sum into arithmetic progressions modulo q gives

$$\begin{aligned}f(\alpha) &= \sum_{x\le X} e\left(\alpha x^k\right)\\ &= \sum_{r=1}^{q} \sum_{0\le j\le \frac{X-r}{q}} e\left(\alpha(qj+r)^k\right)\\ &= \sum_{r=1}^{q} e\left(ar^k/q\right) \sum_{0\le j\le \frac{X-r}{q}} e\left(\beta(qj+r)^k\right).\end{aligned}$$

Now

$$e\left(\beta(qj+r)^k\right) - \frac{1}{q}\int_{qj}^{q(j+1)} e\left(\beta\gamma^k\right) d\gamma = \frac{1}{q}\int_{qj}^{q(j+1)}\left(e\left(\beta(qj+r)^k\right) - e\left(\beta\gamma^k\right)\right) d\gamma.$$

By the mean value theorem

$$\left|e\left(\beta(qj+r)^k\right) - e\left(\beta\gamma^k\right)\right| \le 2\pi k|\beta|\left(q(j+1)\right)^{k-1} q \le 2\pi k|\beta| q(X+q)^{k-1},$$

(the latter holds since $q(j+1) \le X + q$).

Thus

$$f(\alpha) = \frac{1}{q}\sum_{r=1}^{q} e\left(ar^k/q\right)\int_0^{q\left\lfloor\frac{X-r}{q}\right\rfloor+q} e\left(\beta\gamma^k\right) d\gamma + O_k\left(|\beta| q(X+q)^{k-1}\right).$$

Finally, adjusting the range of integration in each summand gives

$$f(\alpha) = \frac{1}{q}\sum_{r=1}^{q} e\left(ar^k/q\right)\int_0^X e\left(\beta\gamma^k\right)d\gamma + O_k\left(q + |\beta| q (X+q)^{k-1}\right). \tag{4.1}$$

Define $S_q(a) := \sum_{r=1}^{q} e\left(ar^k/q\right)$ and $\nu(\beta) := \int_0^X e\left(\beta\gamma^k\right)d\gamma$. Then we have obtained the following lemma.

Lemma 4.1. *Let $\alpha \in \mathbb{T}$, $a \in \mathbb{Z}$ and $q \in \mathbb{N}$, then*

$$f(\alpha) = q^{-1}S_q(a)\nu(\alpha - a/q) + O_k\left(q + |q\alpha - a|(X+q)^{k-1}\right). \tag{4.2}$$

4.2 Definition of the Major Arcs

Let $a, q \in \mathbb{Z}$ with $1 \le a \le q$ and $(a,q) = 1$. We define the major arc centred at a/q, with width parameter w, to be the set

$$\mathfrak{M}_q(a) := \left\{\alpha \in \mathbb{T} : \|\alpha - a/q\| \le \frac{1}{2X^w}\right\}.$$

The set of major arcs up to height parameter h is defined as the union

$$\mathfrak{M} := \bigcup_{\substack{1\le a\le q\le X^h\\(a,q)=1}} \mathfrak{M}_q(a).$$

We would like to ensure the major arcs are disjoint. Suppose $\mathfrak{M}_{q_1}(a_1) \cap \mathfrak{M}_{q_2}(a_2) \ne \emptyset$. Then it follows that

$$\left\|\frac{a_1}{q_1} - \frac{a_2}{q_2}\right\| \le \frac{1}{X^w}. \tag{4.3}$$

It can be checked that for integers $a \in \mathbb{Z}$ and $q \in \mathbb{N}$

$$\left\|\frac{a}{q}\right\| = \frac{|l|}{q},$$

where l is a residue of a, modulo q, with minimal absolute value. First suppose $q_1 \ne q_2$. Then $q_1 q_2 < X^{2h}$. Hence if $h \le w/2$ we have

$$\left\|\frac{a_1q_2 - a_2q_1}{q_1q_2}\right\| = \left\|\frac{a_1}{q_1} - \frac{a_2}{q_2}\right\| < \frac{1}{q_1q_2},$$

and thus $a_1q_1 - a_2q_2 \equiv 0 \mod q_1q_2$. However, this implies q_1 divides q_2 and vice-versa, a contradiction.

Therefore, assuming $h \le w/2$, $q_1 = q_2 = q$. But then because $h < w$

$$\left\|\frac{a_1 - a_2}{q}\right\| < \frac{1}{q},$$

which must mean $a_1 \equiv a_2 \mod q$, and therefore $a_1 = a_2$. Hence to ensure the major arcs are disjoint it suffices to assume $h \le w/2$, which we will do from now on.

Suppose $\alpha \in \mathfrak{M}_q(a)$. Let $g_{q,a}(\alpha) := q^{-1}S_q(a)\nu(\alpha - a/q)$. We'd like to estimate

$$f(\alpha)^s - g_{q,a}(\alpha)^s.$$

Since the above function is defined on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and hence periodic on $\mathbb{R}$, we may choose our representative α to ensure $\|\alpha - a/q\| = |\alpha - a/q|$. Then $|q\alpha - a| \le q\|\alpha - a/q\| \le \frac{1}{2}X^{h-w}$. Both $f(\alpha)$ and $g_{q,a}(\alpha)$ are trivially bounded above by X. Therefore by (4.2)

$$\begin{aligned} f(\alpha)^s - g_{q,a}(\alpha)^s &= \left(f(\alpha) - g_{q,a}(\alpha)\right)\left(f(\alpha)^{s-1} + f(\alpha)^{s-2}g_{q,a}(\alpha) + \cdots + g_{q,a}(\alpha)^{s-1}\right) \\ &\ll_s |f(\alpha) - g_{q,a}(\alpha)|\, X^{s-1} \\ &\ll_{s,k} \left(q + |q\alpha - a|\,(X+q)^{k-1}\right)X^{s-1} \\ &\ll_{s,k} X^{s+h-1} + X^{(s+h-1)+(k-1-w)} \\ &\ll_{s,k} X^{(s+h-1)+\max\{0,k-1-w\}}. \end{aligned}$$

Integrating, we have

$$\begin{aligned}\int_{\mathfrak{M}_q(a)} f(\alpha)^s e(-\alpha n)d\alpha &= \int_{\mathfrak{M}_q(a)} g_{q,a}(\alpha)^s e(-\alpha n)d\alpha + O_{s,k}\left(X^{s+h-w-1+\max\{0,k-1-w\}}\right)\\ &= q^{-s}S_q(a)^s e(-an/q)\int_{\frac{a}{q}-\frac{1}{2}X^{-w}}^{\frac{a}{q}+\frac{1}{2}X^{-w}} \nu(\alpha-a/q)^s e\big(-(\alpha-a/q)n\big)d\alpha\\ &\quad + O_{s,k}\left(X^{s+h-w-1+\max\{0,k-1-w\}}\right)\\ &= q^{-s}S_q(a)^s e(-an/q)\int_{-\frac{1}{2}X^{-w}}^{\frac{1}{2}X^{-w}} \nu(\beta)^s e(-\beta n)d\beta\\ &\quad + O_{s,k}\left(X^{s+h-w-1+\max\{0,k-1-w\}}\right).\end{aligned}$$

Let $J_s(n;Q) := \int_{-Q}^{Q} \nu(\beta)^s e(-\beta n)d\beta$ and

$$\mathfrak{S}_s(n;Q) = \sum_{\substack{1\le a\le q\le Q\\(a,q)=1}} q^{-s}S_q(a)^s e(-an/q).$$

Then by disjointness of the major arcs (assuming $h \le w/2$)

$$\int_{\mathfrak{M}} f(\alpha)^s e(-\alpha n)d\alpha = \mathfrak{S}_s\left(n;X^h\right) J_s\left(n;\tfrac{1}{2}X^{-w}\right) + O_{s,k}\left(X^{s+3h-w-1+\max\{0,k-1-w\}}\right), \qquad (4.4)$$

(here we've used the trivial estimate that there are at most X^{2h} coprime pairs (a,q) with $1 \le a \le q \le X^h$).

Next we'd like to show that we can extend to infinity the range of summation and integration for $\mathfrak{S}_s(n;Q)$ and $J_s(n;Q)$, without introducing too much error into the formula (4.4). First we'll deal with $J_s(n;Q)$. Clearly we require a bound on $\nu(\beta)$. The next lemma provides this and is a consequence of the change of variables formula from elementary calculus. Somewhat embarrassingly, I had forgotten how to do this. As penance I therefore made myself prove the following lemma in tedious detail.

Lemma 4.2. *For all $\beta \in \mathbb{R}$*

$$\nu(\beta) \ll_k \frac{X}{\left(1+X^k|\beta|\right)^{1/k}}.$$

Proof. We first suppose $\beta > 0$. Here is a reminder (for myself) of the change of variables formula: If $f : [c,d] \to \mathbb{C}$ is continuous and $g : [a,b] \to [c,d]$ continuously differentiable then

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b f\big(g(t)\big)g'(t)dt. \qquad (4.5)$$

For our first application of this result set $f_1(\gamma) = e\left(\beta\gamma^k\right)$ $(\gamma \in [0,X])$ and $g_1(\xi) = X\xi$ $(\xi \in [0,1])$. Then

$$\nu(\beta) = \int_{g_1(0)}^{g_1(1)} f_1(\gamma)d\gamma = \int_0^1 f_1\big(g_1(\xi)\big)g_1'(\xi)d\xi \qquad (4.6)$$

$$= X\int_0^1 e\left(\beta X^k\xi^k\right)d\xi. \qquad (4.7)$$

For our next application, take $f_2(\xi) = e\left(\beta X^k\xi^k\right)$ $(\xi \in [0,1])$ and $g_2(t) = \frac{1}{X}(t/\beta)^{1/k}$ $(t \in [\varepsilon,\beta], \varepsilon > 0)$. Then $g_2'(t) = \frac{1}{Xkt^{1-1/k}\beta^{1/k}}$ and so

$$\begin{aligned}\int_{g_2(\varepsilon)}^1 e\left(\beta X^k\xi^k\right)d\xi &= \int_{g_2(\varepsilon)}^{g_2(\beta X^k)} f_2(\xi)d\xi\\ &= \int_{\varepsilon}^{\beta X^k} f_2\big(g_2(t)\big)g_2'(t)dt\\ &= \frac{1}{Xk\beta^{1/k}}\int_{\varepsilon}^{\beta X^k}\frac{e(t)}{t^{1-1/k}}dt.\end{aligned}$$

Since $g_2(\varepsilon) \to 0$ as $\varepsilon \to 0+$, an easy application of the dominated convergence theorem gives us that

$$\begin{aligned}\nu(\beta) &= X \lim_{\varepsilon \to 0+} \int_{g_2(\varepsilon)}^{1} e\left(\beta X^k \xi^k\right) d\xi \\ &= \frac{1}{k\beta^{1/k}} \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{\beta X^k} \frac{e(t)}{t^{1-1/k}} dt.\end{aligned}$$

By the monotone convergence theorem, for any $x > 0$

$$\begin{aligned}\int_{(0,x]} \frac{1}{t^{1-1/k}} dt &= \lim_{n \to \infty} \int_{\frac{1}{n}}^{x} \frac{1}{t^{1-1/k}} dt \\ &= \lim_{n \to \infty} \left(kx^{1/k} - \frac{k}{n^{1/k}} \right) = kx^{1/k} < \infty.\end{aligned}$$

Hence by the dominated convergence theorem, with majorant $\frac{1}{t^{1-1/k}}$, it follows that $\frac{e(t)}{t^{1-1/k}}$ is absolutely integrable on $(0, x]$ and we have

$$\int_{(0,x]} \frac{e(t)}{t^{1-1/k}} dt = \lim_{\varepsilon \to 0+} \int_{\varepsilon}^{x} \frac{e(t)}{t^{1-1/k}} dt.$$

and

$$\lim_{x \to 0+} \int_{(0,x]} \frac{e(t)}{t^{1-1/k}} dt \ll \lim_{x \to 0+} x^{1/k} = 0.$$

Thus if $f(x) := \int_{(0,x]} \frac{e(t)}{t^{1-1/k}} dt$

$$\nu(\beta) \ll_k |\beta|^{-1/k} \sup_{x>0} |f(x)|.$$

If $y > x > 0$

$$|f(y) - f(x)| \leq k\left(y^{1/k} - x^{1/k}\right).$$

So f is continuous on $(0, \infty)$. Also $f(x) \to 0$ as $x \to 0+$, whence it follows that f is bounded on any interval of the form $(0, B]$, and that this bound is independent of β (clearly the only variable it depends on is k). We will show that $L := \lim_{x \to \infty} f(x)$ exists, and thus $|f(x)| \leq |L| + 1$ for all large enough x. Consequently, $\sup_{x>0} |f(x)|$ is an absolute constant depending only on k.

By definition

$$f(x) = \int_0^x \frac{e(t)}{t^{1-1/k}} dt = \int_0^x \frac{\cos(2\pi t)}{t^{1-1/k}} dt + i \int_0^x \frac{\sin(2\pi t)}{t^{1-1/k}} dt.$$

$$\begin{aligned}\int_0^x \frac{\sin(2\pi t)}{t^{1-1/k}} dt &= \sum_{j=0}^{\lfloor 2x \rfloor} \int_{j/2}^{(j+1)/2} \frac{\sin(2\pi t)}{t^{1-1/k}} dt - \int_x^{(\lfloor 2x \rfloor + 1)/2} \frac{\sin(2\pi t)}{t^{1-1/k}} dt \\ &= \sum_{j=0}^{\lfloor 2x \rfloor} (-1)^j \phi_j - \int_x^{(\lfloor 2x \rfloor + 1)/2} \frac{\sin(2\pi t)}{t^{1-1/k}} dt,\end{aligned}$$

where

$$\phi_j := \int_{j/2}^{(j+1)/2} \frac{|\sin(2\pi t)|}{t^{1-1/k}} dt = \int_0^{1/2} \frac{\sin(2\pi u)}{(u + j/2)^{1-1/k}} dt$$

is a decreasing sequence of non-negative reals.

$$\left| \int_x^{(\lfloor 2x \rfloor + 1)/2} \frac{\sin(2\pi t)}{t^{1-1/k}} dt \right| \leq \frac{1}{2x^{1-1/k}} \to 0$$

as $x \to \infty$. Similarly

$$\phi_j \leq \frac{1}{2}(j/2)^{1/k-1} \to 0$$

as $j \to \infty$. Hence

$$\lim_{x \to \infty} \int_0^x \frac{\sin(2\pi t)}{t^{1-1/k}} dt = \sum_{j=0}^{\infty} (-1)^j \phi_j,$$

which converges by the alternating series test. A similar argument shows

$$\lim_{x\to\infty}\int_0^x \frac{\cos(2\pi t)}{t^{1-1/k}}dt$$

exists. We have established that for all $\beta > 0$

$$\nu(\beta) \ll_k |\beta|^{-1/k}.$$

If $\beta < 0$ the same inequality holds since $\nu(\beta) = \overline{\nu(-\beta)}$. Suppose $|\beta| \le X^{-k}$. Then trivially

$$|\nu(\beta)| \le X = 2^{1/k}\frac{X}{(1+1)^{1/k}} \le 2^{1/k}\frac{X}{(1+X^k|\beta|)^{1/k}}.$$

Next suppose $|\beta| > X^{-k}$. Then

$$\nu(\beta) \ll_k \frac{1}{|\beta|^{1/k}} = 2^{1/k}\frac{X}{(X^k|\beta|+X^k|\beta|)^{1/k}} \le 2^{1/k}\frac{X}{(1+X^k|\beta|)^{1/k}}.$$

□

Corollary 4.3. *Provided $s > k$, the integral*

$$J_s(n) := \int_{\mathbb{R}} \nu(\beta)^s e(-\beta n) d\beta$$

exists and is absolutely convergent. Moreover,

$$|J_s(n) - J_s(n;Q)| \ll_k \frac{1}{Q^{\frac{s}{k}-1}}$$

and

$$J_s(n) \ll_k X^{s-k}.$$

In particular

$$J_s\left(n; \tfrac{1}{2}X^{-w}\right) = J_s(n) + O_{s,k}\left(X^{w(\frac{s}{k}-1)}\right).$$

Proof. For $Q > 0$

$$\begin{aligned}|J_s(n) - J_s(n;Q)| &\ll_k \int_Q^\infty \beta^{-s/k} d\beta \\ &\ll_{k,s} \frac{1}{Q^{\frac{s}{k}-1}}.\end{aligned}$$

Also the trivial estimate for $\nu(\beta)$ gives

$$J_s\left(n; X^{-k}\right) \ll X^{s-k}.$$

Combining this with

$$J_s(n) - J_s\left(n; X^{-k}\right) \ll_k \frac{1}{X^{-k(\frac{s}{k}-1)}},$$

gives the required estimate for $J_s(n)$. □

Next we would like to extend to infinity the range of summation of $\mathfrak{S}_s(n;Q)$. This time we require a bound on $S_q(a)$, however this is easily furnished by Weyl's inequality.

Lemma 4.4. *Suppose $s > 2^k$. Then the* singular series

$$\mathfrak{S}_s(n) := \sum_{\substack{q=1 \\ 1\le a\le q \\ (a,q)=1}}^{\infty} q^{-s} S_q(a)^s e(-an/q),$$

is absolutely convergent and bounded by some constant independent of n (but dependent on s and k). Moreover, if $\varepsilon \in \left(0, \frac{1}{s2^{k-1}}\right)$ and $\delta := s\left(\frac{1}{2^{k-1}} - \varepsilon\right) - 2 > 0$, then

$$|\mathfrak{S}_s(n) - \mathfrak{S}_s(n; Q)| \ll_{\varepsilon,k,s} \frac{1}{Q^\delta}.$$

In particular, if we take $\varepsilon := \frac{1}{s2^k}$ then we can guarantee $\delta \geq \frac{1}{2^k}$.

Proof. By Weyl's inequality applied to the polynomial $p(x) = \frac{a}{q}x^k$, we have

$$S_q(a) = \sum_{r=1}^{q} e\left(ar^k/q\right) \ll_{\varepsilon,k} q^{1+\varepsilon}\left(q^{-1} + q^{-1} + q^{1-k}\right)^{2^{1-k}} \ll_{\varepsilon,k} q^{1+\varepsilon-2^{1-k}}.$$

So

$$\left(q^{-1}S_q(a)\right)^s e(-an/q) \ll_{\varepsilon,k,s} q^{s\left(\varepsilon-2^{1-k}\right)}.$$

Combining this with the trivial estimate $\phi(q) \leq q$, we have that for fixed q

$$\sum_{\substack{1\leq a\leq q\\(a,q)=1}} \left(q^{-1}S_q(a)\right)^s e(-an/q) \ll_{\varepsilon,k,s} q^{1-s\left(2^{1-k}-\varepsilon\right)}.$$

The absolute convergence of $\mathfrak{S}_s(n)$ follows provided

$$s\left(2^{1-k} - \varepsilon\right) - 1 > 1. \tag{4.8}$$

Moreover if (4.8) holds, then we have the bound (uniform in n)

$$\mathfrak{S}_s(n) \ll_{\varepsilon,k,s} \sum_{q=1}^{\infty} \frac{1}{q^{s\left(2^{1-k}-\varepsilon\right)}} = O_{\varepsilon,s,k}(1).$$

Although it is not sufficient, it is necessary for (4.8) that $s2^{1-k} > 2$, i.e. $s \geq 2^k + 1$. Can we get away with this being sufficient? We want to find some ε for which

$$s\left(2^{1-k} - \varepsilon\right) > 2.$$

This holds (remembering $s \geq 2^k + 1$) if

$$2^{1-k} > s\varepsilon.$$

Hence we may take any $\varepsilon \in \left(0, \frac{1}{s2^{k-1}}\right)$, e.g. $\varepsilon = \frac{1}{s2^k}$. Setting $\delta := s\left(2^{1-k} - \varepsilon\right) - 2 > 0$, this also gives the estimate

$$\begin{aligned}\mathfrak{S}_s(n) - \mathfrak{S}_s(n) &\ll_{\varepsilon,k,s} \sum_{q>Q} \frac{1}{q^{\delta+1}}\\ &\leq \int_{\lfloor Q\rfloor}^{\infty} \frac{1}{t^{\delta+1}}dt\\ &\ll_{\varepsilon,k,s} \frac{1}{Q^\delta}.\end{aligned}$$

An easy calculation shows $\delta \geq \frac{1}{2^k}$ if $\varepsilon = \frac{1}{s2^k}$. □

Using these estimates we have that for $0 < h \leq w/2$, $s > k$ and $s > 2^k$

$$\begin{aligned}\mathfrak{S}_s\left(n; X^h\right) J_s\left(n; \tfrac{1}{2}X^{-w}\right) &= \mathfrak{S}_s(n) J_s(n) + O_{s,k}\left(X^{\frac{w}{k}(s-k)}\right) + O_{s,k}\left(X^{s-k-\frac{h}{2^k}}\right) + O_{s,k}\left(X^{\frac{w}{k}(s-k)-\frac{h}{2^k}}\right)\\ &= \mathfrak{S}_s(n) J_s(n) + O_{s,k}\left(X^{\frac{w}{k}(s-k)} + X^{s-k-\frac{h}{2^k}}\right).\end{aligned}$$

Combining this with (4.4) gives

$$\int_{\mathfrak{M}} f(\alpha)^s e(-\alpha n)d\alpha - \mathfrak{S}_s(n) J_s(n) \ll_{s,k} X^{\frac{w}{k}(s-k)} + X^{s-k-\frac{h}{2^k}} + X^{s+3h-w-1+\max\{0,k-1-w\}}. \tag{4.9}$$

As mentioned at the start of the notes, if we take $X = n^{1/k}$ then we hope to establish that

$$\int_{\mathfrak{M}} f(\alpha)^s e(-\alpha n) d\alpha = C_{s,k}(n) n^{s/k-1} + o\left(n^{s/k-1}\right),$$

where $C_{s,k}(n)$ is a non-negative function of n satisfying $1 \ll_{s,k} C_{s,k}(n) \ll_{s,k} 1$. We won't yet show that we can take $C_{s,k} n^{s/k-1} = \mathfrak{S}_s(n) J_s(n)$, but we will show the error term in (4.9) is $o(X^{s-k})$ (as we would hope). In order to do this it is clear that it suffices to ensure the following five inequalities are all satisfied (we are assuming $k \geq 3$):

(i) $0 < h \leq w/2$.

(ii) $s > 2^k$.

(iii) $\frac{w}{k}(s-k) < s-k$.

(iv) $s - k - \frac{h}{2^k} < s - k$.

(v) $s + 3h - w - 1 + \max\{0, k-1-w\} < s - k$.

A little calculation shows this set of inequalities is equivalent to

(a) $h > 0$.

(b) $s > 2^k$.

(c) $k - 1 + 3h < w < k$.

We can therefore take (for example) $h := \frac{1}{5}$ and $w := k - \frac{1}{5}$. However, I'm not yet sure if these values for our height and width parameters will suffice for our minor arc estimate. We have however obtained the following lemma.

Lemma 4.5. *Suppose $s > 2^k$ and let $\mathfrak{M}$ denote the major arcs with height parameter $h \in (0, 1/3)$ and width parameter $w = k - 1 + 3h + \delta$ with $\delta \in (0, 1-3h)$. Then*

$$\int_{\mathfrak{M}} f(\alpha)^s e(-\alpha n) d\alpha - \mathfrak{S}_s(n) J_s(n) \ll_{s,k} X^{s-k-\sigma}, \tag{4.10}$$

where

$$\sigma := \min\left\{\tfrac{h}{2^k},\ 1 - 3h - \delta\right\}.$$

Proof. Substituting the formula for w into (4.9) gives (4.10) with

$$\sigma := \min\left\{\left(\tfrac{s}{k} - 1\right)(1 - 3h - \delta),\ \tfrac{h}{2^k},\ 1 - 3h - \delta\right\}.$$

Since $s > 2^k$ and $k \geq 3$ one can check that $\frac{s}{k} - 1 \geq 1$. □

When $X = n^{1/k}$, let

$$R_s^*(n) := \int_{\mathfrak{M}} f(\alpha)^s e(-\alpha n) d\alpha$$

and

$$\mathfrak{J}_s(n) := \frac{J_s(n)}{n^{\frac{s}{k}-1}}.$$

Then we have the following corollary.

Corollary 4.6. *With the same assumptions as in Lemma 4.5*

$$R_s^*(n) = \mathfrak{S}_s(n) \mathfrak{J}_s(n) n^{\frac{s}{k}-1} + O_{s,k}\left(n^{\frac{s}{k}-1-\frac{\sigma}{k}}\right),$$

where

$$\sigma := \min\left\{\tfrac{h}{2^k},\ 1 - 3h - \delta\right\}$$

and

$$\mathfrak{S}_s(n) \mathfrak{J}_s(n) = O_{s,k}(1).$$

Proof. Lemma 4.4 tells us

$$\mathfrak{S}_s(n) = O_{s,k}(1).$$

and by Corollary 4.3

$$\mathfrak{J}_s(n) = \frac{J_s(n)}{n^{\frac{s}{k}-1}} = O_k(1).$$

□

We call $\mathfrak{J}_s(n)$ the *singular integral*. Corollary 4.6 goes part of the way to establishing our claim that $\mathfrak{S}_s(n)J_s(n) = C_{s,k}(n)n^{s/k-1}$ where $1 \ll C_{s,k}(n) \ll 1$. There is in fact a very useful arithmetic interpretation of the function $C_{s,k}$ which allows us to deduce some conditions which ensure it is positive. However, this interpretation is based on techniques from multiplicative number theory and multivariable calculus which I don't need to get to grips with at the moment.

5 Putting it all together

Now we've established some restrictions on the height and width of our major arcs, we can establish a suitable minor arc estimate. Throughout this section we will assume $X = n^{1/k}$, $s > 2^k$, $h \in (0, 1/3)$ and $w = k - 1 + 3h + \delta$ with $\delta \in (0, 1 - 3h)$.

Let $\alpha \in \mathfrak{m} := \mathbb{T} \setminus \mathfrak{M}$. By Dirichlet's theorem on Diophantine approximation, there exists $a, q \in \mathbb{Z}$ with $(a, q) = 1$ and $1 \leq q \leq 2X^w$ such that $\|\alpha - a/q\| \leq (q2X^w)^{-1} \leq (2X^w)^{-1}$. Since $\alpha \notin \mathfrak{M}$, we must have $q > X^h$. Hence by Weyl's inequality

$$\begin{aligned} f(\alpha) &\ll_{\varepsilon,k} X^{1+\varepsilon} \left(X^{-h} + X^{-1} + X^{w-k} \right)^{2^{1-k}} \\ &\ll_{\varepsilon,k,h,\delta} X^{1+\varepsilon - 2^{1-k} \min\{h,\ 1-3h-\delta\}}. \end{aligned}$$

Let $\theta := 2^{1-k} \min\{h,\ 1 - 3h - \delta\} - \varepsilon$. Combining this with Hua's Lemma we have

$$\begin{aligned} \left| \int_{\mathfrak{m}} f(\alpha)^s e(-\alpha n) d\alpha \right| &\leq \sup_{\alpha \in \mathfrak{m}} |f(\alpha)|^{s-2^k} \int_{\mathbb{T}} |f(\alpha)|^{2^k} d\alpha \\ &\ll_{\varepsilon,k,s,h,\delta} X^{\left(s-2^k\right)(1-\theta)+2^k-k+\varepsilon} \\ &= X^{s-k-\left(s-2^k\right)\theta+\varepsilon} \\ &\leq X^{s-k-(\theta-\varepsilon)}. \end{aligned}$$

Taking $h = \delta = 1/5$ and ε sufficiently small, we obtain

$$R_s(n) - R_s^*(n) \ll_{s,k} X^{s-k-\frac{1}{2^{k+1}}} = n^{\frac{s}{k}-1-\frac{1}{k2^{k+1}}}. \tag{5.1}$$

Recalling Corollary 4.6 and noting that with $h = \delta = 5$ we have $\sigma = \frac{1}{52^k} \geq \frac{1}{k2^{k+1}}$, we've finally established:

Theorem 5.1. *Let $s > 2^k$. Then there exists $\tau > 0$ (in fact we can take $\tau = \frac{1}{k^2 2^{k+1}}$) such that*

$$R_s(n) = \mathfrak{S}_s(n)\mathfrak{J}_s(n)n^{\frac{s}{k}-1} + O_{s,k}\left(n^{\frac{s}{k}-1-\tau}\right),$$

where both the singular series

$$\mathfrak{S}_s(n) := \sum_{\substack{q=1 \\ 1\leq a\leq q \\ (a,q)=1}}^{\infty} q^{-s} S_q(a)^s e(-an/q)$$

and the singular integral

$$\mathfrak{J}_s(n) := n^{-\frac{s}{k}+1} \int_{\mathbb{R}} \nu(\beta)^s e(-\beta n) d\beta.$$

are absolutely convergent and of the order $O_{s,k}(1)$ (uniform in n).

This does not establish that $G(k) \leq 2^k + 1$. To do so one performs two calculations. First, use multivariable calculus to show

$$\mathfrak{J}_s(n) = \frac{\Gamma\left(1 + 1/k\right)^s}{\Gamma\left(s/k\right)},$$

where Γ is the classical gamma function. It can therefore be deduced that $\mathfrak{J}_s(n)$ is in fact an absolute *positive* constant independent of n. Second, use facts about Gauss sums from multiplicative number theory to show $\mathfrak{S}_s(n) \gg 1$.

I may at some stage add these calculations to this set of notes.

References

[Bro07] T. D. Browning. Analytic number theory lecture notes. Available at `http://www.maths.bris.ac.uk/~sp2331/ANT_notes.pdf`, 2007.

[Dav05] H. Davenport. *Analytic methods for Diophantine equations and Diophantine inequalities*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2005. With a foreword by R. C. Vaughan, D. R. Heath-Brown and D. E. Freeman, Edited and prepared for publication by T. D. Browning.

[Gow] W. T. Gowers. Additive and combinatorial number theory lecture notes. Available at `http://www.dpmms.cam.ac.uk/~wtg10/addnoth.notes.dvi`.

[Gre07] B. J. Green. Additive number theory lecture notes. Available at `http://www.dpmms.cam.ac.uk/~bjg23/ANT.html`, 2007.

[Vau97] R. C. Vaughan. *The Hardy-Littlewood method*, volume 125 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, second edition, 1997.

[Woo01] T. D. Wooley. Analytic methods for Diophantine problems, unpublished lecture notes. 2001.