Littlewood's conjecture on exponential sums

Let $\mathbb{T}$ denote $\mathbb{R}$ mod 2π. We are interested in functions on $\mathbb{T}$ of the form

$$f(\theta) = e^{ia_1\theta} + e^{ia_2\theta} + \cdots + e^{ia_n\theta} \tag{1}$$

where $a_1 < a_2 < \cdots < a_n$ are distinct integers (positive, negative, or zero). We have $f(0) = n$, and the L^2 norm of f is $\sqrt{n}$. A conjecture credited to Littlewood [2], concerning the L^1 norm of a function f of the form (1), was finally proved in 1981 [3].

Theorem 1 *There exists an absolute positive constant C such that if f is any function of the form (1), then*

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|\, d\theta \geq C \log n.$$

We will follow the proofs appearing in [3] and in Section 9.6 of [1]. Before proving Theorem 1, we give some definitions and facts about Fourier series.

We let $L^2(\mathbb{T})$ denote the set of all square-integrable functions on $\mathbb{T}$ (that is, functions $\phi : \mathbb{T} \to \mathbb{C}$ such that $\int_0^{2\pi} |\phi(\theta)|^2\, d\theta$ is finite). Technically, the elements of $L^2(\mathbb{T})$ are equivalence classes of functions (two functions are equivalent if they agree almost everywhere) but there are certain subtleties we will not explore in detail.

For $\phi \in L^2(\mathbb{T})$, we define the Fourier coefficients

$$\widehat{\phi}(k) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\theta) e^{-ik\theta} d\theta$$

where $k \in \{\ldots, -2, -1, 0, 1, 2, \ldots\}$. If we let ℓ^2 denote the set of two-sided sequences $\{c_k\}_{-\infty}^{\infty}$ of complex numbers such that $\sum_{-\infty}^{\infty} |c_k|^2$ converges, we then have a one-to-one correspondence between $L^2(\mathbb{T})$ and ℓ^2.

That is, given $\phi \in L^2(\mathbb{T})$, its Fourier coefficients satisfy $\sum_{-\infty}^{\infty} |\widehat{\phi}(k)|^2 < \infty$, and given complex numbers $\{c_k\}_{-\infty}^{\infty}$ satisfying $\sum_{-\infty}^{\infty} |c_k|^2 < \infty$, the series $\sum_{-\infty}^{\infty} c_k e^{ik\theta}$ converges in the L^2 norm to a function in $L^2(\mathbb{T})$.

We define the usual inner product on $L^2(\mathbb{T})$:

$$\langle \phi, \psi \rangle = \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta)\overline{\psi(\theta)} d\theta.$$

The norm induced by this inner product is the usual L^2 norm:

$$\sqrt{\langle \phi, \phi \rangle} = \|\phi\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |\phi(\theta)|^2 \, d\theta \right)^{1/2}.$$

We have the Cauchy-Schwarz inequality,

$$|\langle \phi, \psi \rangle| \leq \|\phi\|_2 \, \|\psi\|_2 \,,$$

and we can also express the inner product and norm in terms of the Fourier coefficients:

$$\langle \phi, \psi \rangle = \sum_{-\infty}^{\infty} c_k \overline{d_k}, \tag{2}$$

$$\|\phi\|_2 = \left(\sum_{-\infty}^{\infty} |c_k|^2 \right)^{1/2}, \tag{3}$$

where $c_k = \widehat{\phi}(k)$ and $d_k = \widehat{\psi}(k)$. Also useful to us is a fact about Fourier coefficients of products of functions. Observe that

$$\widehat{\phi\psi}(k) = \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta)\psi(\theta) e^{-ik\theta} d\theta = \langle \phi(\theta), \overline{\psi(\theta)} e^{ik\theta} \rangle$$

so, by Cauchy-Schwarz, we have

$$|\widehat{\phi\psi}(k)| \leq \|\phi(\theta)\|_2 \|\overline{\psi(\theta)} e^{ik\theta}\|_2$$

or equivalently,

$$|\widehat{\phi\psi}(k)| \leq \|\phi\|_2 \, \|\psi\|_2 \,. \tag{4}$$

In general, we say $\phi \in L^2(\mathbb{T})$ has "nonpositive support" if $\widehat{\phi}(k) = 0$ for all positive k (loosely, if ϕ is a combination of nonpositive powers of $e^{i\theta}$). We note that if ϕ and ψ have nonpositive support, then $\phi+\psi$, $\phi\psi$, e^{ϕ}, and $e^{-\phi}$ all have nonpositive support. To see this, observe that if we have

$$\begin{aligned} \phi(\theta) &= c_0 + c_1 e^{-i\theta} + c_2 e^{-i2\theta} + \cdots \\ \psi(\theta) &= d_0 + d_1 e^{-i\theta} + d_2 e^{-i2\theta} + \cdots \end{aligned}$$

then we have

$$\begin{aligned} \phi(\theta) + \psi(\theta) &= (c_0{+}d_0) + (c_1{+}d_1)e^{-i\theta} + (c_2{+}d_2)e^{-i2\theta} + \cdots \\ \phi(\theta)\psi(\theta) &= (c_0 d_0) + (c_0 d_1{+}c_1 d_0)e^{-i\theta} + (c_0 d_2{+}c_1 d_1{+}c_2 d_0)e^{-i2\theta} + \cdots \end{aligned}$$

and we also have

$$\begin{aligned} e^{\phi} &= 1 + \frac{\phi}{1!} + \frac{\phi^2}{2!} + \frac{\phi^3}{3!} + \cdots \\ e^{-\phi} &= 1 - \frac{\phi}{1!} + \frac{\phi^2}{2!} - \frac{\phi^3}{3!} + \cdots \end{aligned}$$

both of which are sums of functions with nonpositive support. (To state these results precisely involves convergence questions which we do not discuss in detail.) We now state and prove some lemmas. One fact we will use is that $\mathrm{Re}(z) \geq 0$ implies $|e^{-z}| \leq 1$ (because $|e^{-x-iy}| = |e^{-x}e^{-iy}| = |e^{-x}| = e^{-x}$).

Lemma 2 *Suppose $g \in L^2(\mathbb{T})$ is real-valued and nonnegative. Then there exists $h \in L^2(\mathbb{T})$ such that:*

- $\mathrm{Re}(h) = g$,
- *h has nonpositive support,*
- *e^{-h} has nonpositive support,*
- $\left|e^{-h}\right| \leq 1$,
- $\|h\|_2 \leq \sqrt{2}\,\|g\|_2$.

Proof. Let $g(\theta) = \sum_{-\infty}^{\infty} c_k e^{ik\theta}$. Since g is real-valued, it follows that $c_k = c_{-k}$. We have

$$\begin{aligned} g(\theta) &= \cdots + c_2 e^{-i2\theta} + c_1 e^{-i\theta} + c_0 + c_1 e^{i\theta} + c_2 e^{i2\theta} + \cdots \\ &= c_0 + 2c_1 \cos\theta + 2c_2 \cos 2\theta + \cdots \\ &= c_0 + 2c_1 \cos(-\theta) + 2c_2 \cos(-2\theta) + \cdots . \end{aligned}$$

If we choose

$$h(\theta) = c_0 + 2c_1 e^{-i\theta} + 2c_2 e^{-i2\theta} + \cdots$$

then $\mathrm{Re}(h) = g$, and h has nonpositive support. It then follows from an earlier remark that e^{-h} has nonpositive support, and it follows from another earlier remark that $\left|e^{-h}\right| \leq 1$. Finally, we note that

$$\begin{aligned} \|h\|_2^2 &= |c_0|^2 + 4\,|c_1|^2 + 4\,|c_2|^2 + \cdots \\ &\leq 2(|c_0|^2 + 2\,|c_1|^2 + 2\,|c_2|^2 + \cdots) = 2\,\|g\|_2^2 . \end{aligned}$$

This completes the proof of Lemma 2.

Lemma 3 *If w is a complex number with $\mathrm{Re}(w) \geq 0$, then $|e^{-w} - 1| \leq |w|$.*

Proof. Consider the integral

$$\int_\gamma -e^{-z} dz \tag{5}$$

where γ is the line segment from $z = 0$ to $z = w$. We can parametrize γ by $z = wt$ where $0 \leq t \leq 1$. Then $dz = w\,dt$, and we have

$$\int_\gamma -e^{-z} dz = \int_0^1 -e^{-wt} w\,dt = \left[e^{-wt}\right]_{t=0}^{t=1} = e^{-w} - 1.$$

So the absolute value of the integral (5) is $|e^{-w} - 1|$. But also, the integrand satisfies $|-e^{-z}| \leq 1$ since $\mathrm{Re}(z) \geq 0$, and we are integrating over a line segment of length $|w|$. This completes the proof of Lemma 3.

Corollary. If $h \in L^2(\mathbb{T})$ satisfies $\mathrm{Re}(h) \geq 0$, then $\left\|e^{-h} - 1\right\|_2 \leq \|h\|_2$.

Proof.

$$\left\|e^{-h} - 1\right\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} \left|e^{-h(\theta)} - 1\right|^2 d\theta \leq \frac{1}{2\pi}\int_0^{2\pi} |h(\theta)|^2 \, d\theta = \|h\|_2^2 .$$

Proof of Theorem 1. As in the statement of the theorem, let

$$f(\theta) = e^{ia_1\theta} + \cdots + e^{ia_n\theta}$$

with integers $a_1 < \cdots < a_n$. Choose m so that

$$\frac{4^{m+1} - 1}{4 - 1} = 1 + 4 + 4^2 + \cdots + 4^m \leq n < 1 + 4 + 4^2 + \cdots + 4^{m+1} = \frac{4^{m+2} - 1}{4 - 1}$$

which implies $m = \lfloor \log(3n + 1)/\log 4 \rfloor - 1$. Then define

$$\begin{aligned} S_0 &= \{a_1\} \\ S_1 &= \{a_2, \ldots, a_5\}, \quad \text{so } |S_1| = 4 \\ S_2 &= \{a_6, \ldots, a_{21}\}, \quad \text{so } |S_2| = 16 \end{aligned}$$

and in general, if $S_{j-1} = \{a_k, \ldots, a_\ell\}$, we define $S_j = \{a_{\ell+1}, \ldots, a_{\ell+4^j}\}$ for $j \leq m$ (so $|S_j| = 4^j$). Also, let $T = \{a_1, \ldots, a_n\} \setminus (S_0 \cup \cdots \cup S_m)$. Define

$$\begin{aligned} q_0(\theta) &= e^{ia_1\theta} \\ q_1(\theta) &= \frac{1}{4}\left(e^{ia_2\theta} + \cdots + e^{ia_5\theta}\right) \\ q_2(\theta) &= \frac{1}{16}\left(e^{ia_6\theta} + \cdots + e^{ia_{21}\theta}\right) \end{aligned}$$

and in general, $q_j(\theta) = \frac{1}{4^j}\sum_{a \in S_j} e^{ia\theta}$, for all $j \leq m$. We note that $|q_j(\theta)| \leq 1$, and it is routine to verify that $\|q_j\|_2 = 1/2^j$.

Here is a sketch of the proof. We will construct a function ϕ that satisfies $|\phi(\theta)| \leq 1$ and whose Fourier coefficients satisfy the following. If $a \in T$, we will have $\widehat{\phi}(a) = 0$, and if $a \in S_0 \cup \cdots \cup S_m$, then $\widehat{\phi}(a)$ will be "close" to $\widehat{\psi}(a)$, where

$$\psi(\theta) = \frac{1}{5}\Big(q_0(\theta) + q_1(\theta) + q_2(\theta) + \cdots + q_m(\theta)\Big). \tag{6}$$

Using (2), we would then have

$$\frac{1}{2\pi}\int_0^{2\pi} \phi(\theta)\overline{f(\theta)}d\theta = \langle \phi, f\rangle = \sum_{j=0}^{m}\sum_{a\in S_j} \widehat{\phi}(a)\overline{\widehat{f}(a)} = \sum_{j=0}^{m}\sum_{a\in S_j} \widehat{\phi}(a) \tag{7}$$

but also

$$\left|\frac{1}{2\pi}\int_0^{2\pi} \phi(\theta)\overline{f(\theta)}d\theta\right| \le \frac{1}{2\pi}\int_0^{2\pi} |\phi(\theta)||\overline{f(\theta)}|d\theta \le \frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|\, d\theta,$$

implying

$$\frac{1}{2\pi}\int_0^{2\pi} |f(\theta)|\, d\theta \ge \left|\sum_{j=0}^{m}\sum_{a\in S_j} \widehat{\phi}(a)\right|. \tag{8}$$

Note that we have not yet used the assumption that $\widehat{\phi}(a)$ is "close" to $\widehat{\psi}(a)$. So we have shown that (8) holds for any ϕ that satisfies $|\phi| \le 1$ and satisfies $\widehat{\phi}(a) = 0$ for $a \in T$.

If we furthermore have $\widehat{\phi}(a)$ "close" to $\widehat{\psi}(a)$, then in (8), we would have

$$\sum_{j=0}^{m}\sum_{a\in S_j} \widehat{\phi}(a) \approx \sum_{j=0}^{m}\frac{1}{5}\sum_{a\in S_j}\frac{1}{4^j} = \frac{1}{5}(m+1) \sim C\log n.$$

What remains, of course, is to actually construct ϕ, as well as to quantify how "close" we can make $\widehat{\phi}(a)$ to $\widehat{\psi}(a)$.

To this end, for each $j \in \{1, \ldots, m\}$, we let $h_j(\theta)$ be the function obtained from $g(\theta) = \frac{1}{4}\,|q_j(\theta)|$ as in Lemma 2. That is, we have

- $\mathrm{Re}(h_j) = \frac{1}{4}\,|q_j|$,

- h_j has nonpositive support,
- e^{-h_j} has nonpositive support,
- $\left|e^{-h_j}\right| \leq 1$,
- $\|h_j\|_2 \leq \sqrt{2} \cdot \frac{1}{4}\|q_j\|_2 = \sqrt{2} \cdot \frac{1}{2^{j+2}}$.

We then define $\phi_0 = \frac{1}{5}q_0$, and for $j \in \{1, \ldots, m\}$, we define

$$\phi_j = \phi_{j-1} \cdot e^{-h_j} + \frac{1}{5}q_j. \tag{9}$$

We can prove by induction that $|\phi_j| \leq 1$ for each j. Notice $|\phi_0| = \frac{1}{5}|q_0| = \frac{1}{5}$. Assuming $|\phi_{j-1}| \leq 1$, then (9) implies

$$\begin{aligned}
|\phi_j| &\leq |\phi_{j-1}|\left|e^{-h_j}\right| + \frac{1}{5}|q_j| \\
&\leq \left|e^{-h_j}\right| + \frac{1}{5}|q_j| \\
&= e^{-\mathrm{Re}(h_j)} + \frac{1}{5}|q_j| \\
&= e^{-|q_j|/4} + \frac{1}{5}|q_j|.
\end{aligned}$$

This is bounded above by 1 because the inequality $e^{-x/4} + \frac{1}{5}x \leq 1$ holds for all $x \in [0, 1]$. Now, from (9), we have

$$\begin{aligned}
\phi_0 &= \frac{1}{5}q_0 \\
\phi_1 &= \frac{1}{5}\Big(q_0 e^{-h_1} + q_1\Big) \\
\phi_2 &= \frac{1}{5}\Big(q_0 e^{-h_1-h_2} + q_1 e^{-h_2} + q_2\Big) \\
&\vdots \\
\phi_m &= \frac{1}{5}\Big(q_0 e^{-h_1-\cdots-h_m} + q_1 e^{-h_2-\cdots-h_m} + \cdots + q_{m-1}e^{-h_m} + q_m\Big)
\end{aligned}$$

Note that each function of the form $e^{-h_j-\cdots-h_m}$ has nonpositive support. It follows that $q_{j-1}e^{-h_j-\cdots-h_m}$ is the product of q_{j-1} and a combination of nonpositive powers of $e^{i\theta}$. But all powers $e^{ia\theta}$ appearing in the Fourier series for q_{j-1} satisfy $a \in S_{j-1}$. This implies that all powers $e^{ia\theta}$ appearing in the Fourier series for $q_{j-1}e^{-h_j-\cdots-h_m}$ satisfy $a \notin S_j \cup \cdots \cup S_m \cup T$.

We now take $\phi = \phi_m$. Note that if $a \in T$, then $\widehat{\phi}(a) = 0$. We now consider $a \in S_j$ for some $j \in \{0, \ldots, m\}$ and try to show that $\widehat{\phi}(a)$ is "close" to $\widehat{\psi}(a)$, where ψ is as defined in (6).

For $a \in S_j$, our earlier remarks imply that the coefficient of $e^{ia\theta}$ in $\phi = \phi_m$ is the same as the coefficient of $e^{ia\theta}$ in

$$p_1 := \frac{1}{5}\Big(q_j e^{-h_{j+1}-\cdots-h_m} + q_{j+1}e^{-h_{j+2}-\cdots-h_m} + \cdots + q_{m-1}e^{-h_m} + q_m\Big).$$

We want to show that this is "close" to the same as the coefficient of $e^{ia\theta}$ in $\frac{1}{5}q_j$, or equivalently, the coefficient of $e^{ia\theta}$ in

$$p_2 := \frac{1}{5}\Big(q_j + q_{j+1} + \cdots + q_{m-1} + q_m\Big).$$

Observe that we have

$$p_1 - p_2 = \frac{1}{5}\Big(q_j(e^{-h_{j+1}-\cdots-h_m}-1) + q_{j+1}(e^{-h_{j+2}-\cdots-h_m}-1) + \cdots + q_{m-1}(e^{-h_m}-1)\Big)$$

and we also have

$$\begin{aligned}\widehat{p_1}(a) - \widehat{p_2}(a) = \widehat{p_1 - p_2}(a) = \frac{1}{5}\Big(&\{q_j(e^{-h_{j+1}-\cdots-h_m} - 1)\}\widehat{\ }(a)\\ &+ \{q_{j+1}(e^{-h_{j+2}-\cdots-h_m} - 1)\}\widehat{\ }(a)\\ &+ \cdots\\ &+ \{q_{m-1}(e^{-h_m} - 1)\}\widehat{\ }(a)\Big)\end{aligned}$$

and therefore

$$|\widehat{p_1}(a) - \widehat{p_2}(a)| \le \frac{1}{5}\Big(\big|\{q_j(e^{-h_{j+1}-\cdots-h_m} - 1)\}\widehat{\ }(a)\big| \\ + \big|\{q_{j+1}(e^{-h_{j+2}-\cdots-h_m} - 1)\}\widehat{\ }(a)\big| \\ + \cdots \\ + \big|\{q_{m-1}(e^{-h_m} - 1)\}\widehat{\ }(a)\big| \Big).$$

We now observe that if $\ell \in \{j, \ldots, m-1\}$, then (4) implies

$$\big|\{q_\ell(e^{-h_{\ell+1}-\cdots-h_m} - 1)\}\widehat{\ }(a)\big| \le \|q_\ell\|_2 \left\|e^{-h_{\ell+1}-\cdots-h_m} - 1\right\|_2 \\ \le \frac{1}{2^\ell} \|h_{\ell+1} + \cdots + h_m\|_2$$

where we have used the corollary to Lemma 3. Continuing, we have

$$\big|\{q_\ell(e^{-h_{\ell+1}-\cdots-h_m} - 1)\}\widehat{\ }(a)\big| \le \frac{1}{2^\ell}\Big(\|h_{\ell+1}\|_2 + \cdots + \|h_m\|_2 \Big) \\ \le \frac{1}{2^\ell}\Big(\frac{\sqrt{2}}{2^{\ell+3}} + \cdots + \frac{\sqrt{2}}{2^{m+2}}\Big) \\ < \frac{1}{2^\ell}\Big(\frac{\sqrt{2}}{2^{\ell+2}}\Big) = \frac{\sqrt{2}}{4} \cdot \frac{1}{4^\ell}.$$

It follows that we have

$$|\widehat{p_1}(a) - \widehat{p_2}(a)| < \frac{1}{5}\Big(\frac{\sqrt{2}}{4} \cdot \frac{1}{4^j} + \frac{\sqrt{2}}{4} \cdot \frac{1}{4^{j+1}} + \cdots + \frac{\sqrt{2}}{4} \cdot \frac{1}{4^{m-1}}\Big) \\ = \frac{\sqrt{2}}{20}\Big(\frac{1}{4^j} + \frac{1}{4^{j+1}} + \cdots + \frac{1}{4^{m-1}}\Big) < \frac{\sqrt{2}}{20} \cdot \frac{4}{3} \cdot \frac{1}{4^j}$$

That is, we have $|\widehat{p_1}(a) - \widehat{p_2}(a)| < (\sqrt{2}/15) \cdot (1/4^j)$. We also have $\widehat{p_2}(a) = (1/5) \cdot (1/4^j)$. It follows that $\widehat{p_1}(a)$ is a complex number of the form

$$\widehat{p_1}(a) = \frac{1}{5} \cdot \frac{1}{4^j} + \delta_a$$

where δ_a is a complex number satisfying $|\delta_a| < (\sqrt{2}/15) \cdot (1/4^j)$. That is, we have shown that for $a \in S_j$, we have

$$\widehat{\phi}(a) = \frac{1}{5} \cdot \frac{1}{4^j} + \delta_a$$

where δ_a is as described above. Applying this to (8), we note that we have

$$\sum_{a\in S_j} \widehat{\phi}(a) = \sum_{a\in S_j} \left(\frac{1}{5}\cdot\frac{1}{4^j} + \delta_a\right) = \frac{1}{5} + \sum_{a\in S_j} \delta_a$$

and we note that $\varepsilon_j := \sum_{a\in S_j} \delta_a$ is a complex number whose modulus is bounded above by $4^j \cdot (\sqrt{2}/15)\cdot(1/4^j) = \sqrt{2}/15$. We then have

$$\sum_{j=0}^{m}\sum_{a\in S_j} \widehat{\phi}(a) = \sum_{j=0}^{m}\left(\frac{1}{5} + \varepsilon_j\right) = \frac{1}{5}(m+1) + \sum_{j=0}^{m}\varepsilon_j = \frac{3}{15}(m+1) + \sum_{j=0}^{m}\varepsilon_j$$

and we note that $\sum_{j=0}^{m}\varepsilon_j$ is a complex number satisfying

$$\left|\sum_{j=0}^{m}\varepsilon_j\right| \leq \frac{\sqrt{2}}{15}(m+1).$$

It follows that $\sum_{j=0}^{m}\sum_{a\in S_j}\widehat{\phi}(a)$ is a complex number satisfying

$$\frac{3-\sqrt{2}}{15}(m+1) \leq \left|\sum_{j=0}^{m}\sum_{a\in S_j}\widehat{\phi}(a)\right| \leq \frac{3+\sqrt{2}}{15}(m+1)$$

From (8), it then follows that $\frac{1}{2\pi}\int_0^{2\pi}|f(\theta)|\,d\theta$ is bounded below by a constant multiple of $\log n$.

References

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