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THE BANACH–TARSKI PARADOX

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In this exposition we clarify the meaning of and prove the following "paradoxical" theorem which was set forth by Stefan Banach and Alfred Tarski in 1924 [**1**]. We were inspired to do this by a recent paper of A. M. Bruckner and Jack Ceder [**2**], where this theorem, among others, is brought into their interesting discussion of the phenomenon of nonmeasurable sets. We are grateful to Professor R. B. Burckel for calling this paper to our attention. We warmly recommend it to the reader. It is our intention here to present a strictly elementary account of this remarkable fact that will be accessible to readers with very little mathematical background. We do presume a little matrix theory and the elements of real analysis. We first state the main theorem and then give precise definitions before launching into its proof. We may as well admit in advance that its proof depends on Zermelo's Axiom of Choice, which is used in a very obvious way in the proof of Theorem C below (the set C selected there is *not* specified in a finitely constructable way).

BANACH–TARSKI THEOREM. *If X and Y are bounded subsets of $\mathbf{R}^3$ having nonempty interiors, then there exist a natural number n and partitions $\{X_j : 1 \leqslant j \leqslant n\}$ and $\{Y_j : 1 \leqslant j \leqslant n\}$ of X and Y, respectively (into n pieces each), such that X_j is congruent to Y_j for all j.*

Loosely speaking, the theorem says that if X and Y are any two objects in space that are each small enough to be contained in some (perhaps very large) ball and each large enough to contain some (perhaps very small) ball, then one can divide X into some finite number of pieces and then reassemble them (using only rigid motions) to form Y. This seems to be patently false if we submit to the foolish practice of confusing the "ideal" objects of geometry with the "real" objects of the world around us. It certainly does seem to be folly to claim that a billiard ball can be chopped into pieces which can then be put back together to form a life-size statue of Banach. We, of course, make no such claim. Even in the world of mathematics, the theorem is astonishing, but true.

DEFINITIONS. For $x=(x_1,x_2,x_3)$ in $\mathbf{R}^3$ we define the *norm* of x to be the number $|x|=(x_1^2+x_2^2+x_3^2)^{1/2}$. The *closed ball* of *radius* $r>0$ *centered* at $a \in \mathbf{R}^3$ is the set $\{x \in \mathbf{R}^3 : |x-a| \leqslant r\}$. A subset X of $\mathbf{R}^3$ is *bounded* if it is contained in some such ball, and X has *nonvoid interior*, if it contains some such ball. An *orthogonal matrix* is a square matrix with real entries whose transpose is also its inverse (its product with its transpose is the identity matrix). By a *rotation* we shall mean a 3×3 orthogonal matrix ρ whose determinant is equal to 1. We also regard such a ρ as a mapping of $\mathbf{R}^3$ onto $\mathbf{R}^3$ by writing $\rho(x)$ for the vector obtained by multiplying ρ by the column vector $x : \rho(x)=y=(y_1,y_2,y_3)$ where $x=(x_1,x_2,x_3)$,

$$\rho=\begin{pmatrix}\rho_{11} & \rho_{12} & \rho_{13}\\ \rho_{21} & \rho_{22} & \rho_{23}\\ \rho_{31} & \rho_{32} & \rho_{33}\end{pmatrix}, \qquad y_i=\sum_{j=1}^{3}\rho_{ij}x_j$$

for $i=1,2,3$. A *rigid motion* (or *Euclidean transformation*) is a mapping r of $\mathbf{R}^3$ onto $\mathbf{R}^3$ having the form $r(x)=\rho(x)+a$ for $x \in \mathbf{R}^3$ where ρ is a fixed rotation and $a \in \mathbf{R}^3$ is fixed. We denote the 3×3

Karl Stromberg received his Ph.D. at the University of Washington in 1958 under the direction of Edwin Hewitt, with whom he is the coauthor of *Real and Abstract Analysis* (Springer-Verlag, 1965). He served on the faculty of the University of Oregon 1960–68 and has been Professor of Mathematics at Kansas State University since 1968. He has also spent single years on the visiting faculty at four other universities: Yale (1958–59), Chicago (1959–60), Uppsala (Sweden) (1966–67), York (England) (1974–75). His main research interests are in Analysis, both classical and abstract. He is the author of the forthcoming book *Classical Real Analysis* (Prindle, Weber, and Schmidt, 1979).—*Editors* identity matrix by ι. Two subsets X and Y of $\mathbf{R}^3$ are said to be congruent and we write $X \cong Y$ if there exists some rigid motion r for which $r(X) = Y$. (Here, as usual, $r(X)$ denotes the set $\{r(x) : x \in X\}$.) By a *partition* of a set X we mean a family of sets whose union is X and any two members of which are either identical or disjoint. Thus, to say that $\{X_j : 1 \leqslant j \leqslant n\}$ is a partition of X into n subsets means that

$$X = X_1 \cup X_2 \cup \cdots \cup X_n \quad \text{and} \quad X_i \cap X_j = \phi \text{ if } i \neq j.$$

It is allowed that some or all X_j be void.

The geometrical significance of our purely algebraic definition of a rotation is perhaps clarified by the next proposition.

PROPOSITION. *Let ρ be a rotation. Then we have the following.*

(i) *The image ρ of any line is a line: $\rho(b+tc) = \rho(b) + t\rho(c)$ for all $b, c \in \mathbf{R}^3$ and $t \in \mathbf{R}$.*

(ii) *Inner products are preserved by ρ; if $x, x' \in \mathbf{R}^3$, $\rho(x) = y$ and $\rho(x') = y'$, then*

$$\sum_{i=1}^{3} y_i y_i' = \sum_{j=1}^{3} x_j x_j'.$$

(iii) *Distances are preserved by ρ: if $x \in \mathbf{R}^3$, then $|\rho(x)| = |x|$.*

(iv) *If $\rho \neq \iota$, then the set $A = \{x \in \mathbf{R}^3 : \rho(x) = x\}$ is a line through the origin: there is a p in $\mathbf{R}^3$ such that $A = \{tp : t \in \mathbf{R}\}$ and $|p| = 1$. We call A the axis of ρ.*

(v) *If q is any point of $\mathbf{R}^3$ having the two properties of p in* (iv), *then $q = p$* or *$q = -p$. We call p and $-p$ the poles of ρ.*

Proof. Assertion (i) is obvious and (iii) follows from (ii) by taking $x' = x$. To prove (v) notice that if $\{tp : t \in \mathbf{R}\} = \{tq : t \in \mathbf{R}\}$ and $|q| = |p| = 1$, then $q = tp$ for some t and $t^2 = t^2|p|^2 = |tp|^2 = |q|^2 = 1$ so t is 1 or -1.

To prove (ii), use the fact that ρ is orthogonal $\left[\sum_i \rho_{ij}\rho_{ik} = \iota_{jk} = 1 \text{ or } 0 \text{ according as } j = k \text{ or } j \neq k\right]$ to write

$$\begin{aligned}
\sum_i y_i y_i' &= \sum_i \Big(\sum_j \rho_{ij} x_j\Big)\Big(\sum_k \rho_{ik} x_k'\Big) \\
&= \sum_i \Big(\sum_j \sum_k \rho_{ij}\rho_{ik} x_j x_k'\Big) \\
&= \sum_j \sum_k \iota_{jk} x_j x_k' = \sum_j x_j x_j'.
\end{aligned}$$

To prove (iv) we need a modest amount of matrix theory and real analysis. The characteristic polynomial $f(\lambda) = \det(\rho - \lambda\iota)$ of ρ is a cubic polynomial having real coefficients so the Intermediate Value Theorem assures that it has at least one real root. Let $\lambda_1, \lambda_2, \lambda_3$ be its three (complex) roots (counting multiplicity) where λ_1 is the largest real root. Then $f(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)$ so

$$\lambda_1\lambda_2\lambda_3 = f(0) = \det\rho = 1. \tag{*}$$

If λ_k is a real root, then the system of equations

$$\sum_{j=1}^{3} (\rho_{ij} - \lambda_k \iota_{ij}) x_j = 0 \qquad (i = 1, 2, 3)$$

has a *real* solution x_1, x_2, x_3 (not all 0) so there is an $x \in \mathbf{R}^3$, $|x| \neq 0$, such that $\rho(x) = \lambda_k x$ from which (iii) yields $|\lambda_k| = 1$, and so $\lambda_k = 1$ or -1. If λ_2 is not real, then λ_3 is its complex conjugate and (*) becomes $\lambda_1|\lambda_2|^2 = 1$ so $\lambda_1 = 1$. If λ_2 is real, then so is λ_3, all three roots are 1 or -1, and (*) shows that $\lambda_1 = 1$ and $\lambda_2 = \lambda_3$. Since $\lambda_1 = 1$, we can take $k = 1$ in the above system to find a vector $p \in \mathbf{R}^3$ with $|p| = 1$ such that $\rho(p) = p$. Then $tp \in A$ for all $t \in \mathbf{R}$. Our job is to see that there are no other vectors in A. Assume that there is a $u \in A$ with $u \neq tp$ for all $t \in \mathbf{R}$. Choose a nonzero vector v that is perpendicular to the plane containing p, u, and 0; that is, $\sum v_j p_j = \sum v_j u_j = 0$. Since $\rho(p) = p$ and $\rho(u) = u$, it follows from (ii) that $\rho(v)$ is also perpendicular to this plane and thence from (iii) that $\rho(v) = v$ or $-v$. Any vector $x \in \mathbf{R}^3$ can be written as $x = \alpha p + \beta u + \gamma v$ for appropriate $\alpha, \beta, \gamma \in \mathbf{R}$ and, by (i), $\rho(x) = \alpha p + \beta u + \gamma \rho(v)$. Since $\rho \neq \iota$, we cannot have $\rho(v) = v$. Therefore $\rho(v) = -v$. The matrix

$$\sigma = \begin{pmatrix} p_1 & u_1 & v_1 \\ p_2 & u_2 & v_2 \\ p_3 & u_3 & v_3 \end{pmatrix}$$

has *nonzero* determinant (because p, u, and v are linearly independent) and the matrix product

$$\rho\sigma = \begin{pmatrix} p_1 & u_1 & -v_1 \\ p_2 & u_2 & -v_2 \\ p_3 & u_3 & -v_3 \end{pmatrix}$$

satisfies

$$-\det\sigma = \det(\rho\sigma) = (\det\rho)(\det\sigma) = \det\sigma$$

so $\det\sigma = 0$. This contradiction completes the proof of (iv). ■

We now prove several theorems and lemmas which are of considerable interest in themselves as well as being vital stepping stones toward our main goal. The first three of these, of which Theorem C is the real key to our story, were set forth by Felix Hausdorff in 1914 [**4**, pp. 469–472]. We consider the two rotations

$$\psi = \begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\phi = \begin{pmatrix} -\cos\theta & 0 & \sin\theta \\ 0 & -1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

where θ is a fixed real number, to be chosen later. (Geometrically, ψ rotates $\mathbf{R}^3$ by 120° about the z-axis and ϕ rotates $\mathbf{R}^3$ by 180° about the line in the xz-plane whose equation is $x\cos\frac{1}{2}\theta = z\sin\frac{1}{2}\theta$.) One checks that the matrix ψ^2 is the same as the matrix ψ except that $\sqrt{3}$ is replaced by $-\sqrt{3}$ and that

$$\psi^3 = \phi^2 = \iota \tag{1}$$

where ι is the identity matrix. Now let G denote the set of all matrices that can be obtained as a product of a finite number of (matrix) factors, each of which is ϕ or ψ. Because of (1), it is clear that G is a group under matrix multiplication (if $\rho, \sigma \in G$, then ρ^{-1}, $\rho\sigma \in G$) and that each $\rho \neq \iota$ in G can be expressed in *at least* one way as a product

$$\rho = \sigma_1 \sigma_2 \cdots \sigma_n \tag{2}$$

where $n \geqslant 1$, each σ_j is ϕ or ψ or ψ^2, and if $1 \leqslant j < n$, then exactly one of σ_j and σ_{j+1} is ϕ. We call such expressions *reduced words* in the *letters* ϕ, ψ, and ψ^2. For example, the expression $\phi\psi^2\phi\phi\psi^2\phi$ is *not* a reduced word because of the two adjacent ϕ's, but it is equal to the reduced word $\phi\psi\phi$ ("equal" means that these products are the same matrix). Thus each element of G other than ι, ϕ, ψ, and ψ^2 can be expressed in at least one of the four forms

$$\alpha = \psi^{p_1}\phi\psi^{p_2}\phi\cdots\psi^{p_m}\phi, \qquad \beta = \phi\psi^{p_1}\phi\psi^{p_2}\cdots\phi\psi^{p_m},$$
$$\gamma = \phi\psi^{p_1}\phi\psi^{p_2}\cdots\phi\psi^{p_m}\phi, \qquad \delta = \psi^{p_1}\phi\psi^{p_2}\phi\cdots\phi\psi^{p_m} \tag{3}$$

where $m \geqslant 1$ and each exponent p_j is 1 or 2 (for $\delta, m > 1$). These are the reduced words having more than one letter. Depending on our choice of θ, it may happen that two reduced words that appear to be different are actually equal; i.e., when multiplied out, they equal the same matrix. For example, if we choose $\theta = \pi$, one checks that $\psi\phi = \phi\psi^2$ and $\psi\phi\psi\phi = \iota$. However, we do have the following remarkable theorem which is the key to our later results.

THEOREM A. [4]. *If* $\cos\theta$ *is a transcendental number, then each element of* G *other than* ι *has exactly one expression as a reduced word in the letters* ϕ, ψ, *and* ψ^2. *That is, if*

$$\text{(i)} \quad \sigma_1\sigma_2\cdots\sigma_n = \rho_1\rho_2\cdots\rho_m$$

where each side of this equation is a reduced word, then $m = n$ and $\sigma_j = \rho_j$ for $1 \leqslant j \leqslant n$.

Proof. We need only show that no reduced word is equal to ι, for then if (i) held true with n as small as possible it would follow that $n = 1$ and $\rho_1 = \sigma_1$.

We first show that if α is as in (3), then $\alpha \neq \iota$. We have $\alpha = \sigma_m\sigma_{m-1}\cdots\sigma_2\sigma_1$ where each σ is either $\psi\phi$ or $\psi^2\phi$. That is, each σ is one of the two matrices

$$\sigma = \begin{bmatrix} \frac{1}{2}\cos\theta & \pm\frac{\sqrt{3}}{2} & -\frac{1}{2}\sin\theta \\ \mp\frac{\sqrt{3}}{2}\cos\theta & \frac{1}{2} & \pm\frac{\sqrt{3}}{2}\sin\theta \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

One checks by induction on m that if $K = (0,0,1)$, then $\sigma_m\sigma_{m-1}\cdots\sigma_1(K) = (\sin\theta P_{m-1}(\cos\theta), \sqrt{3}\sin\theta Q_{m-1}(\cos\theta), R_m(\cos\theta))$ where the P, Q, and R are certain polynomials with rational coefficients, their subscripts are their degrees, and their leading coefficients are

$$-\frac{1}{2}\left(\frac{3}{2}\right)^{m-1}, \pm\frac{1}{2}\left(\frac{3}{2}\right)^{m-1}, \left(\frac{3}{2}\right)^{m-1},$$

respectively. In fact, simple computations show that

$$\begin{aligned} P_0(x) &= -\frac{1}{2}, Q_0(x) = \pm\frac{1}{2}, R_1(x) = x, \\ P_m(x) &= \frac{1}{2}xP_{m-1}(x) \pm \frac{3}{2}Q_{m-1}(x) - \frac{1}{2}R_m(x) \\ Q_m(x) &= \mp\frac{1}{2}xP_{m-1}(x) + \frac{1}{2}Q_{m-1}(x) \pm \frac{1}{2}R_m(x) \\ R_{m+1}(x) &= (1-x^2)P_{m-1}(x) + xR_m(x). \end{aligned}$$

This done, we see that since $\cos\theta$ is a root of no polynomial with rational coefficients, it is impossible that $\alpha(K) = K$ (else $R_m(\cos\theta) - 1 = 0$) and so $\alpha \neq \iota$.

Now we see that no β as in (3) can equal ι, for otherwise $\alpha = \phi\beta\phi = \phi\iota\phi = \phi^2 = \iota$. Similarly, if $\gamma = \iota$, then $\delta = \phi\gamma\phi = \iota$, so it remains only to rule out the possibility that $\delta = \iota$.

Assume that $\delta = \iota$ where δ is as in (3) and m is the smallest natural number for which this is true. Of course $m > 1$. If $p_1 = p_m$, then $\psi^{p_1+p_m}$ is either ψ^2 or $\psi^4 = \psi$ so

$$\iota = \psi^{-p_1}\delta\psi^{p_1} = \phi\psi^{p_2}\cdots\phi\psi^{p_1+p_m}$$

is a reduced word of the form β which is impossible. Thus $p_1 + p_m = 3$. In case $m > 3$, we have

$$\iota = \phi\psi^{p_m}\delta\psi^{p_1}\phi = \psi^{p_2}\phi\cdots\phi\psi^{p_{m-1}}$$

which is again of the form δ, contrary to the minimality of m. Therefore $m = 2$ or 3. But $m = 2$ yields $\iota = \psi^{p_2}\delta\psi^{p_1} = \phi$ while $m = 3$ yields $\iota = \phi\psi^{p_3}\delta\psi^{p_1}\phi = \psi^{p_2}$ and these results are ridiculous. We conclude that $\delta = \iota$ is impossible. ∎

We hereby choose and fix any θ such that $\cos\theta$ is transcendental. Of course all but countably many real numbers θ have this property. (Incidentally, it follows from the Generalized Lindemann Theorem that any nonzero algebraic θ will do; e.g., $\theta=1$.)

If an element $\rho\in G$ is expressed in its unique way as a reduced word as in (2), we call n the *length* of ρ and we say that σ_1 is the *first letter* of ρ or that ρ *begins with* σ_1. We write $l(\rho)=n$ and $l(\iota)=0$.

As usual, by a *partition* of a set X, we mean a pairwise disjoint family of subsets of X whose union is X.

THEOREM B. *There exists a partition* $\{G_1,G_2,G_3\}$ *of* G *into three nonvoid subsets such that for each* ρ *in* G *we have*

(i) $\rho\in G_1\Leftrightarrow\phi\rho\in G_2\cup G_3$,
(ii) $\rho\in G_1\Leftrightarrow\psi\rho\in G_2$,
(iii) $\rho\in G_1\Leftrightarrow\psi^2\rho\in G_3$.

(Note, for example, that $\phi\rho$ need not begin with ϕ. If $\rho=\phi\psi\phi$, then $\phi\rho=\psi\phi$ begins with ψ.)

Proof. Assign the elements of G inductively according to their lengths as follows. Put

$$\iota\in G_1,\phi\in G_2,\psi\in G_2,\psi^2\in G_3. \tag{4}$$

Suppose that $n\geqslant 1$ is some integer such that each $\sigma\in G$ with $l(\sigma)\leqslant n$ has been assigned to exactly one of G_1, G_2, and G_3. We now assign all elements of length $n+1$. If $l(\sigma)=n$ and σ begins with ψ or ψ^2, put

$$\begin{aligned}&\phi\sigma\in G_2 \text{ if } \sigma\in G_1,\\ &\phi\sigma\in G_1 \text{ if } \sigma\in G_2\cup G_3.\end{aligned} \tag{5}$$

If $l(\sigma)=n$ and σ begins with ϕ, put

$$\psi\sigma\in G_{j+1} \text{ if } \sigma\in G_j, \tag{6}$$

$$\psi^2\sigma\in G_{j+2} \text{ if } \sigma\in G_j \tag{7}$$

for $j=1$, 2, 3 where $G_4=G_1$ and $G_5=G_2$. By induction our partition is now formed. The assignment of any element of length n can be easily determined in n steps. For example, if $\rho=\psi\phi\psi\phi\psi^2\phi\psi^2$, then $l(\rho)=7$ and we note successively, beginning with the last letter, that

$$\psi^2\in G_3,\phi\psi^2\in G_1,\psi^2\phi\psi^2\in G_3,\phi\psi^2\phi\psi^2\in G_1$$
$$\psi\phi\psi^2\phi\psi^2\in G_2,\phi\psi\phi\psi^2\phi\psi^2\in G_1,\rho\in G_2.$$

One easily checks that the elements of length two satisfy

$$\{\phi\psi,\phi\psi^2,\psi^2\phi\}\subset G_1,\psi\phi\in G_3,$$

and therefore that (i)–(iii) hold if $l(\rho)\leqslant 1$ (for example, both sides of equivalence (i) are false unless $\rho=\iota$). For an inductive proof of (i)–(iii), suppose that $n>1$ is some integer and that these three equivalences are known to hold for all $\rho\in G$ having $l(\rho)<n$. Now let $\rho\in G$ with $l(\rho)=n$ be given.

CASE 1. Suppose that ρ begins with ϕ. Then (6) and (7), with $\sigma=\rho$, imply (ii) and (iii), respectively. Since $\phi\rho$ has length $n-1$, our induction hypothesis yields

$$\begin{aligned}\rho\notin G_1&\Leftrightarrow\phi(\phi\rho)=\rho\in G_2\cup G_3\\ &\Leftrightarrow\phi\rho\in G_1\Leftrightarrow\phi\rho\notin G_2\cup G_3\end{aligned}$$

and so (i) also holds for ρ.

CASE 2. Suppose that ρ begins with ψ. Then (i) follows from (5) with $\sigma=\rho$. We have $\psi\rho=\psi^2\sigma$ where $l(\sigma)=n-1$ and σ begins with ϕ, so (7) and (6) yield

$$\psi\rho=\psi^2\sigma\in G_2\Leftrightarrow\sigma\in G_3\Leftrightarrow\rho=\psi\sigma\in G_1\Leftrightarrow\psi^2\rho=\sigma\in G_3$$

which proves (ii) and (iii) for ρ.

CASE 3. Suppose that ρ begins with ψ^2. As in Case 2, (i) follows from (5). Here we have $\psi\rho=\sigma$ has length $n-1$ and begins with ϕ. So again (6) and (7) yield

$$\psi\rho=\sigma\in G_2\Leftrightarrow\rho=\psi^2\sigma\in G_1\Leftrightarrow\sigma\in G_2\Leftrightarrow\psi^2\rho=\psi\sigma\in G_3$$

proving (ii) and (iii) in this final case. ■

THEOREM C. *There exists a partition* $\{P,S_1,S_2,S_3\}$ *of the unit sphere* $S=\{x\in\mathbf{R}^3:|x|^2=x_1^2+x_2^2+x_3^2=1\}$ *into four subsets such that*

(i) *P is countable*, (ii) $\phi(S_1)=S_2\cup S_3$,

(iii) $\psi(S_1)=S_2$, (iv) $\psi^2(S_1)=S_3$.

Proof. Let $P=\{p\in S:\rho(p)=p$ for some $\rho\in G$ with $\rho\neq\iota\}$. Since G is countable and each $\rho\neq\iota$ leaves just two points of S fixed (the poles of its axis of rotation) we see that (i) obtains. For each $x\in S\setminus P$, let $G(x)=\{\rho(x):\rho\in G\}$. Each such $G(x)$ is a subset of $S\setminus P$ (if $\rho(x)\in P$ for some ρ, then $\sigma\rho(x)=\rho(x)$ for some $\sigma\neq\iota$ so $\rho^{-1}\sigma\rho(x)=x$, $\rho^{-1}\sigma\rho\neq\iota$, and $x\in P$), $x\in G(x)[x=\iota(x)]$, and any two such sets $G(x)$ and $G(y)$ are either disjoint or identical (if $t\in G(x)\cap G(y)$, say $\rho(x)=t=\sigma(y)$, and $z\in G(x)$, say $z=\tau(x)$, then $z=\tau(x)=\tau\rho^{-1}(t)=\tau\rho^{-1}\sigma(y)\in G(y)$; whence, $G(x)\cap G(y)\neq\phi\Rightarrow G(x)=G(y)$). Therefore, the family of sets $\mathfrak{F}=\{G(x):x\in S\setminus P\}$ is a partition of $S\setminus P$. Next, choose exactly one point from each member of $\mathfrak{F}$ and denote the set of points so chosen by C. The set C has the properties:

$$C\subset S\setminus P, \tag{a}$$

$$c_1\neq c_2 \text{ in } C\Rightarrow G(c_1)\cap G(c_2)=\phi, \tag{b}$$

$$x\in S\setminus P\Rightarrow x\in G(c) \text{ for some } c\in C \tag{c}$$

because $x\in G(c)\Leftrightarrow c\in G(x)$ for all $x,c\in S\setminus P$. Now define

$$S_j=G_j(C)=\{\rho(c):\rho\in G_j,c\in C\}$$

for $j=1,2,3$ where G_1,G_2,G_3 are as in Theorem B. Using (a) and the fact that $G(x)\subset S\setminus P$ if $x\in S\setminus P$, we see that $S_j\subset S\setminus P$ for each j. The fact that $G=G_1\cup G_2\cup G_3$ and (c) imply that $S\setminus P=S_1\cup S_2\cup S_3$. If $j\neq i$ in $\{1,2,3\}$, then $S_j\cap S_i=\phi$ (otherwise, for $x\in S_j\cap S_i$, we have $x=\rho(c_1)=\sigma(c_2)$ for some $c_1,c_2\in C,\rho\in G_j,\sigma\in G_i$ so (b) yields $c_1=c_2=c$, say, and hence $\sigma^{-1}\rho(c)=c$ while $c\notin P$ from which $\sigma^{-1}\rho=\iota$ and $\rho=\sigma$ contrary to $G_j\cap G_i=\phi$). Therefore $\{P,S_1,S_2,S_3\}$ is a partition of S.

Finally, we apply (i)–(iii) of Theorem B to write

$$\phi(S_1)=\{\phi\rho(c):\rho\in G_1,c\in C\}=\{\tau(c):\tau\in G_2\cup G_3,c\in C\}=S_2\cup S_3,$$
$$\psi(S_1)=\{\psi\rho(c):\rho\in G_1,c\in C\}=\{\tau(c):\tau\in G_2,c\in C\}=S_2,$$
$$\psi^2(S_1)=\{\psi^2\rho(c):\rho\in G_1,c\in C\}=\{\tau(c):\tau\in G_3,c\in C\}=S_3$$

which proves (ii)–(iv). ■

The following lemma and its use in deducing Theorems D and E from Theorem C are contributions of W. Sierpiński (see [**6**]).

LEMMA. *If P is any countable subset of S, then there exists a countable set Q and a rotation ω such that $P\subset Q\subset S$ and $\omega(Q)=Q\setminus P$.*

Proof. The idea of the proof is very simple. We first select an axis of rotation for ω that contains no point of P, then we use the countability of $P\times P\times\mathbf{N}$ to select one of the uncountable supply of angles of rotation for ω that make ω satisfy $P\cap\omega^n(P)=\phi$ for all $n\geqslant 1$, and finally we put

$$Q=P\cup\bigcup_{n=1}^{\infty}\omega^n(P) \tag{8}$$

We now give details.

Among all vectors $v=(v_1,v_2,v_3)$ in S having $v_3=0$, there are only countably many for which v or $-v$ is in P. Select any $v=(v_1,v_2,0)\in S$ such that neither v nor $-v$ is in P. Writing $u=(1,0,0)$ and

$$\sigma=\begin{bmatrix} v_1 & v_2 & 0\\ -v_2 & v_1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

we see that σ is a rotation, $\sigma(v)=u$, and the set $\sigma(P)$ contains neither u nor $-u$. For real numbers t, consider the rotations

$$\tau_t=\begin{pmatrix} 1 & 0 & 0\\ 0 & \cos t & -\sin t\\ 0 & \sin t & \cos t\end{pmatrix}$$

that leave u fixed. For each triple (x,y,n) with $x,y\in\sigma(P)$ and $n\in\mathbf{N}$, it follows easily from the fact that $x_2^2+x_3^2>0$ that there exist either exactly n or exactly 0 values of t in $[0,2\pi[$ for which $\tau_t^n(x)=y$ according as $x_1=y_1$ or $x_1\neq y_1$. Since there are only countably many such triples in all, there are only countably many t for which the equality

$$\sigma(P)\cap\bigcup_{n=1}^{\infty}\tau_t^n\sigma(P)=\phi \tag{9}$$

fails. Fix any $t\in\mathbf{R}$ for which (9) obtains and write $\tau=\tau_t$. Now define $\omega=\sigma^{-1}\tau\sigma$ and define Q as in (8). Since $\tau^n\sigma=\sigma\omega^n$ for all n, (9) yields $\sigma(P\cap\omega(Q))=\sigma\left(P\cap\bigcup_{n=1}^{\infty}\omega^n(P)\right)=\phi$ from which we have $P\cap\omega(Q)=\phi$. But $Q=P\cup\omega(Q)$ so the proof is finished. ■

THEOREM D. *There exists a partition $\{T_j:1\leqslant j\leqslant 10\}$ of the unit sphere S into ten (disjoint) subsets and a corresponding set $\{\rho_j:1\leqslant j\leqslant 10\}$ of rotations such that $\{\rho_j(T_j):1\leqslant j\leqslant 6\}$ is a partition of S into six subsets and $\{\rho_j(T_j):7\leqslant j\leqslant 10\}$ is a partition of S into four subsets. Moreover, we can take T_7, T_8, and T_9 to all be rotates of S_1 and take T_1, T_2, T_3, and T_{10} to all be countable.*

Proof. We continue our previous notation and define

$$U_1=\phi(S_2),\ U_2=\psi\phi(S_2),\ U_3=\psi^2\phi(S_2),$$
$$V_1=\phi(S_3),\ V_2=\psi\phi(S_3),\ V_3=\psi^2\phi(S_3).$$

By Theorem C it is clear that $\{U_j,V_j\}$ is a partition of S_j for $j=1,2,3$ and that these six sets along with P form a partition of S into seven subsets. Now let

$$T_7=U_1,\ T_8=U_2,\ T_9=U_3,\ T_{10}=P,$$
$$\rho_7=\psi^2\phi,\ \rho_8=\phi\psi^2,\ \rho_9=\psi\phi\psi,\ \rho_{10}=\iota$$

and check that $\rho_{10}(T_{10})=P$ and $\rho_j(T_j)=S_{j-6}$ for $j=7,8,9$ so that $\{\rho_j(T_j):7\leqslant j\leqslant 10\}$ is indeed a partition of S. We shall now divide $S\backslash(T_7\cup T_8\cup T_9\cup T_{10})=V_1\cup V_2\cup V_3$ into six pieces. Let Q and ω be as in the preceding Lemma and define

$$T_1=\rho_8(S_1\cap Q),\ T_2=\rho_9(S_2\cap Q),\ T_3=\rho_7(S_3\cap Q)$$
$$T_4=\rho_8(S_1\backslash Q),\ T_5=\rho_9(S_2\backslash Q),\ T_6=\rho_7(S_3\backslash Q).$$

Plainly,

$$\{T_1,T_4\}\text{ partitions }\rho_8(S_1)=V_1,$$
$$\{T_2,T_5\}\text{ partitions }\rho_9(S_2)=V_2,$$
$$\{T_3,T_6\}\text{ partitions }\rho_7(S_3)=V_3,$$

and thus we see that $\{T_j:1\leqslant j\leqslant 10\}$ partitions S. Next define

$$\rho_4=\rho_8^{-1},\rho_5=\rho_9^{-1},\rho_6=\rho_7^{-1} \quad \text{and} \quad \rho_j=\omega^{-1}\rho_{j+3}$$

for $j=1,2,3$. Evidently,

$$\rho_{j+3}(T_{j+3})=S_j\setminus Q \qquad (j=1,2,3)$$

and, since $P\subset Q$, the union of these three sets is $S\setminus Q$. Finally, we have

$$\rho_j(T_j)=\omega^{-1}\rho_{j+3}(T_j)=\omega^{-1}(S_j\cap Q) \qquad (j=1,2,3)$$

so these three sets are disjoint and their union is $\omega^{-1}(Q\setminus P)=Q$. ■

If, for a subset T of S, we write $T'=\{tx:x\in T,0<t\leqslant 1\}$, then $S'=\{y\in\mathbf{R}^3:0<|y|\leqslant 1\}$ is the punctured ball obtained from the solid unit ball $B=\{y\in\mathbf{R}^3:|y|\leqslant 1\}$ by removing the origin $O=(0,0,0)$, and it is clear that the first sentence of Theorem D remains true if we replace S by S' and T_j by T_j' throughout. We use this observation in the next proof.

THEOREM E. *There exists a partition* $\{B_k:1\leqslant k\leqslant 40\}$ *of the closed unit ball* B *into forty subsets and a corresponding set* $\{r_k:1\leqslant k\leqslant 40\}$ *of rigid motions such that* $\{r_k(B_k):1\leqslant k\leqslant 24\}$ *partitions* B *into twenty-four subsets and* $\{r_k(B_k):25\leqslant k\leqslant 40\}$ *partitions* B *into sixteen subsets.*

Proof. Apply the above Lemma to the case that P is the singleton set $\{u\}$ where $u=(1,0,0)\in S$ to obtain a countable set Q with $u\in Q\subset S$ and a rotation ρ_0 such that $\rho_0(Q)=Q\setminus\{u\}$. Next let $N_1=\{\frac{1}{2}(q-u):q\in Q\}$ and define the rigid motion r_0 by

$$r_0(x)=\rho_0\big(x+\tfrac{1}{2}u\big)-\tfrac{1}{2}u.$$

Plainly the vector 0 is in N_1 and $r_0(N_1)=N_1\setminus\{0\}$. Writing $N_2=B\setminus N_1$, $s_1=r_0$, $s_2=\iota$, and $M_h=s_h(N_h)$ for $h=1$ and 2, we see that $\{N_1,N_2\}$ partitions B and $\{M_1,M_2\}$ partitions $S'=B\setminus\{0\}$. We complete the proof by combining these partitions and rigid motions with the partition $\{T_j':1\leqslant j\leqslant 10\}$ of S' and the rotations $\{\rho_j:1\leqslant j\leqslant 10\}$ as in the remark following Theorem D.

Notice that, for each $j(1\leqslant j\leqslant 10)$, the family $\{T_j'\cap\rho_j^{-1}(M_i):1\leqslant i\leqslant 2\}$ partitions T_j' and that in turn $\{M_h\cap T_j'\cap\rho_j^{-1}(M_i):1\leqslant h\leqslant 2\}$ partitions $T_j'\cap\rho_j^{-1}(M_i)$ for $i=1$ and $i=2$. Thus $\{M_h\cap T_j'\cap\rho_j^{-1}(M_i):1\leqslant h\leqslant 2,1\leqslant i\leqslant 2,1\leqslant j\leqslant 10\}$ is a partition of S' into forty subsets and the forty sets

$$B_{hij}=s_h^{-1}\big[M_h\cap T_j'\cap\rho_j^{-1}(M_i)\big]$$

form a partition of B while for each fixed j the four sets

$$\rho_j s_h(B_{hij})=M_i\cap\rho_j(M_h\cap T_j') \tag{10}$$

$(1\leqslant h\leqslant 2,1\leqslant i\leqslant 2)$ form a partition of $\rho_j(T_j')$. We now invoke Theorem D to see that the families

$$\{\rho_j s_h(B_{hij}):1\leqslant h\leqslant 2,1\leqslant i\leqslant 2,1\leqslant j\leqslant 6\}$$

$$\{\rho_j s_h(B_{hij}):1\leqslant h\leqslant 2,1\leqslant i\leqslant 2,7\leqslant j\leqslant 10\}$$

are each a partition of S' while, for fixed i, (10) shows that the respective families of twelve and eight sets are each a partition of M_i which we can in turn map to partitions of N_i via s_i^{-1}. Therefore, writing $r_{hij}=s_i^{-1}\rho_j s_h$, we infer that

$$\{r_{hij}(B_{hij}):1\leqslant h\leqslant 2,1\leqslant i\leqslant 2,1\leqslant j\leqslant 6\}$$

and

$$\{r_{hij}(B_{hij}):1\leqslant h\leqslant 2,1\leqslant i\leqslant 2,7\leqslant j\leqslant 10\}$$

are partitions of B into twenty-four sets and sixteen sets, respectively. Finally, relabel the forty sets B_{hij} and the forty rigid motions r_{hij} with single subscripts $k=1,2,\dots,40$. ■

DEFINITION. We shall say that two subsets X and Y of $\mathbf{R}^3$ are *piecewise congruent* and we write $X\sim Y$ if, for some natural number n, there exist a partition $\{X_j:1\leqslant j\leqslant n\}$ of X into n

subsets and a corresponding set $\{f_j : 1 \leqslant j \leqslant n\}$ of rigid motions such that $\{f_j(X_j) : 1 \leqslant j \leqslant n\}$ is a partition of Y. In case X is piecewise congruent to a subset of Y, we shall write $X \lesssim Y$.

Our next theorem gives some simple properties of the relations just defined.

THEOREM F. *For subsets* X, Y, *and* Z *of* $\mathbf{R}^3$ *we have*

(i) $X \sim X$,
(ii) $X \sim Y \Rightarrow Y \sim X$,
(iii) $X \sim Y$ *and* $Y \sim Z \Rightarrow X \sim Z$,
(iv) $X \sim Y \Rightarrow X \lesssim Y$,
(v) $X \lesssim Y$ *and* $Y \lesssim Z \Rightarrow X \lesssim Z$,
(vi) $X \subset Y \Rightarrow X \lesssim Y$,
(vii) $X \lesssim Y$ *and* $Y \lesssim X \Rightarrow X \sim Y$.

Proof. Since $Y \subset Y$, (iv) is banal. Since ι is a rigid motion, (i) and (vi) are obvious (with $n = 1$). Assertion (ii) follows from the fact that inverses of rigid motions are rigid motions.

To prove (v), suppose that $\{X_j : 1 \leqslant j \leqslant n\}$ and $\{Y_i : 1 \leqslant i \leqslant m\}$ are partitions of X and Y, respectively, and that $\{f_j : 1 \leqslant j \leqslant n\}$ and $\{g_i : 1 \leqslant i \leqslant m\}$ are sets of rigid motions such that $\{f_j(X_j) : 1 \leqslant j \leqslant n\}$ is a partition of some $Y_0 \subset Y$ and $\{g_i(Y_i) : 1 \leqslant i \leqslant m\}$ is a partition of some $Z_0 \subset Z$. Then one readily checks that the mn sets $A_{ij} = X_j \cap f_j^{-1}(Y_i)$ form a partition of X (for fixed j, the m sets $A_{1j}, A_{2j}, \ldots, A_{mj}$ are pairwise disjoint and their union is X_j) and, for fixed i, the n sets $f_j(A_{ij}) = Y_i \cap f_j(X_j) (1 \leqslant j \leqslant n)$ form a partition of $Y_i \cap Y_0$ so $\{g_i f_j(A_{ij}) : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ is a pairwise disjoint family whose union is some subset Z_1 of Z. Each composite mapping $g_i f_j$ is a rigid motion, so we have $X \sim Z_1$ and hence $X \lesssim Z$. This proves (v). The same argument proves (iii) by taking $Y_0 = Y$ and $Z_0 = Z$.

To prove (vii), suppose that $X \sim Y_0$ and $Y \sim X_0$ where $Y_0 \subset Y$ and $X_0 \subset X$. Let the notation be as in the preceding paragraph with $X = Z$ and $X_0 = Z_0$. We prove that $X \sim Y$ by copying a well-known proof of the Schröder–Bernstein Theorem. First define f on X and g on Y by $f(x) = f_j(x)$ if $x \in X_j$ and $g(y) = g_i(y)$ if $y \in Y_i$. For $E \subset X$, define $E' \subset X$ by

$$E' = X \setminus g[Y \setminus f(E)]. \tag{11}$$

Plainly,

$$E \subset F \subset X \Rightarrow E' \subset F' \tag{12}$$

Let $\mathfrak{D} = \{E : E \subset X, E \subset E'\}$. Notice that $\phi \in \mathfrak{D}$. Let $D = \bigcup \mathfrak{D}$ be the union of all the sets that belong to $\mathfrak{D}$. For each $E \in \mathfrak{D}$ we have $E' \subset D'$ by (12) so $E \subset D'$. Thus $D \subset D'$ and so (12) yields $D' \subset (D')'$; hence, $D' \in \mathfrak{D}$, $D' \subset D$, and $D' = D$. Put $E = D$ in (11) to obtain

$$D = X \setminus g[Y \setminus f(D)], X \setminus D = g[Y \setminus f(D)].$$

Clearly, $X \setminus D \subset X_0$. Now define, for $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant m$,

$$A_j = D \cap X_j, A_{n+i} = g_i[Y_i \setminus f(D)], h_j = f_j, \quad \text{and} \quad h_{n+i} = g_i^{-1}.$$

It follows that $\{A_1, \ldots, A_n\}$ partition D, $\{A_{n+1}, \ldots, A_{n+m}\}$ partitions $X \setminus D$, $\{h_1(A_1), \ldots, h_n(A_n)\}$ partitions $f(D)$, and $\{h_{n+1}(A_{n+1}), \ldots, h_{n+m}(A_{n+m})\}$ partitions $Y \setminus f(D)$. Therefore $X \sim Y$. ■

Recall that a *closed ball* in $\mathbf{R}^3$ is any set of the form $A = \{x \in \mathbf{R}^3 : |x - a| \leqslant \varepsilon\}$ where $a \in \mathbf{R}^3$ and $\varepsilon > 0$ are given. Recall also that a *translate* of a set $A \subset \mathbf{R}^3$ is any set of the form $A + b = \{x + b : x \in A\}$ where $b \in \mathbf{R}^3$ is given.

THEOREM G. *If* $A \subset \mathbf{R}^3$ *is a closed ball and if* $A_1, A_2, \ldots, A_n$ *are a finite number of translates of* A, *then*

$$A \sim \bigcup_{j=1}^{n} A_j.$$

Proof. We may suppose that $A=\{x\in\mathbf{R}^3:|x|\leqslant\varepsilon\}$ for some $\varepsilon>0$. Choose any $a\in\mathbf{R}^3$ for which $|a|>2\varepsilon$ and let $A'=A+a=\{y\in\mathbf{R}^3:|y-a|\leqslant\varepsilon\}$. We use Theorem E to show that $A\sim(A\cup A')$. So let the B_k and r_k be as in that theorem. For any set $D\subset\mathbf{R}^3$ and any $\delta>0$, let $\delta D=\{\delta x:x\in D\}$. We consider the partition $\{\varepsilon B_k:1\leqslant k\leqslant 40\}$ of A. Define rigid motions s_k by

$$s_k(x)=\varepsilon r_k\left(\frac{1}{\varepsilon}x\right)\quad\text{if } 1\leqslant k\leqslant 24,$$

$$s_k(x)=\varepsilon r_k\left(\frac{1}{\varepsilon}x\right)+a\quad\text{if } 25\leqslant k\leqslant 40.$$

(Note that if r is a rigid motion ($r(x)=\rho(x)+b$ where ρ is a rotation) and $s(x)=\varepsilon r\left(\frac{1}{\varepsilon}x\right)$, then s is a rigid motion because $s(x)=\rho(x)+\varepsilon b$.) From Theorem E we see that $\{s_k(\varepsilon B_k):1\leqslant k\leqslant 24\}$ partitions A, $\{s_k(\varepsilon B_k):25\leqslant k\leqslant 40\}$ partitions A', and so, since $A\cap A'=\phi$. $\{s_k(\varepsilon B_k):1\leqslant k\leqslant 40\}$ partitions $A\cup A'$. This proves that

$$A\sim(A\cup A').$$

We now prove the theorem by induction on n. The theorem is obvious if $n=1$. Suppose that $n>1$ is such that A is piecewise congruent to the union of any $n-1$ of its translates and let $A_1,\ldots,A_n$ by any n of its translates. By hypothesis $A\sim[A_1\cup\cdots\cup A_{n-1}]$ and it is obvious that $A_n\backslash[A_1\cup\cdots\cup A_{n-1}]$ is congruent (by translation) to a subset of A' so we have

$$A_1\cup\cdots\cup A_n\lesssim A\cup A'\sim A.$$

But clearly $A\lesssim A_1\cup\cdots\cup A_n$ so Theorem F yields $A\sim A_1\cup\cdots\cup A_n$. ∎

We now state the Banach–Tarski Theorem again and then prove it.

THEOREM H. *If X and Y are bounded subsets of $\mathbf{R}^3$ having nonvoid interiors, then $X\sim Y$.*

Proof. Choose interior points a and b of X and Y, respectively, and then choose $\varepsilon>0$ such that $A=\{x\in\mathbf{R}^3:|x|\leqslant\varepsilon\}$ satisfies $A+a\subset X$ and $A+b\subset Y$. Since X is bounded, there exist a finite number $A_1,\ldots,A_n$ of translates of A whose union contains X. We therefore have, using Theorem G,

$$A\lesssim X\subset(A_1\cup\cdots\cup A_n)\sim A$$

so it follows from Theorem F that $X\sim A$. Similarly $Y\sim A$. Another application of Theorem F gives $X\sim Y$. ∎

REMARKS. 1. The number 40 that appears in Theorem E is not the smallest possible. In fact, R. M. Robinson showed in 1947 [5] that there is a partition of B into five sets (one of them a singleton) which can be reassembled by rigid motions to form two disjoint closed balls of unit radius. Moreover, T. J. Dekker and J. deGroot proved [3] that these five sets can be chosen so that each is both connected and locally connected.

2. It follows from Theorem C that, since Lebesgue measure λ^3 on $\mathbf{R}^3$ is rotation invariant, none of the three sets $S_k'=\{tx:x\in S_k,0<t<1\}, 1\leqslant k\leqslant 3$, can be Lebesgue measurable.

3. A poor analogue of Theorem C can be explicitly constructed (no Axiom of Choice) in the plane as follows. Fix any transcendental complex number c with $|c|=1$ (plenty of these exist, since there are only countably many z with $|z|=1$ that fail to be transcendental; we can take $c=e^i$). Now let X be the set of all complex numbers of the form

$$z=\sum_{j=0}^{n}a_jc^j$$

where n and $a_0,a_1,\ldots,a_n$ are nonnegative integers. Each $z\in X$ has a unique such expression. Let X_0 be the set of those z for which $a_0=0$ and let $X_1=X\backslash X_0$. Then $\{X_0,X_1\}$ partitions X. Define the rotation ρ of the plane by $\rho(z)=cz$ and define the translation τ by $\tau(z)=z+1$. Then $\rho(X)=X_0$ and $\tau(X)=X_1$ so the sets X_0 and X_1 are each congruent to X. The reason that this analogue is "poor" is twofold: X is both countable and unbounded.

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SEX DIFFERENCES IN MATHEMATICS: HOW *NOT* TO DEAL WITH THEM

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Even casual observation of this distinguished assemblage reveals sex differences among mathematicians. There are both male and female mathematicians! This may seem to be a vehement way of expressing the obvious. But it seems to be not at all obvious to those who portray the history of mathematics. A case in point is an important collection of portraits and biographies of mathematicians throughout the ages on a wall map entitled "Men of Modern Mathematics" [7]. There is a woman among them, Emmy Noether. But absent are other women who, despite enormous obstacles, contributed significantly to mathematics, e.g., Sophia Germain and Sonya Kovalevsky. In a similar vein, a well-known and otherwise excellent textbook on the history of mathematics has no women listed in the name index—and seemingly not mentioned in the text—not even Emmy Noether, although her father, Max Noether, is listed [3]. Still another well-known text on the history of mathematics referred to Hypatia of the fourth century as the first woman mathematician to be mentioned in the history of mathematics—but it referred to no other women, at least not in its first three editions, even as recently as 1969; however, there is a brief reference to Emmy Noether in the most recent edition of the text [5].

These are illustrations of ways in which *not* to deal with sex differences in mathematics. Do not ignore or overlook or hide the achievements of one sex. Let us find out more about these achievements and make them known to our colleagues, our students and the general public.

True, famous women mathematicians throughout history can be counted on one's fingers. But when mathematics students were asked to name such women, they usually did not reach even the first finger. For example, when the request to name famous women mathematicians was made of 26 mathematics majors in a junior-senior level algebra class, 24 did not list any names. In contrast, when they were then asked to name three to five famous mathematicians, 22 students answered, listing an average of four (male) mathematicians. It is important to increase the awareness of the contributions of women mathematicians in the past (cf. [4], [20]).

Nor should we *belittle* the women's contributions. At a recent conference on women in the history of mathematics, one of the participants remarked that on the whole she was disappointed

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