22/11/19

Γενικευμένα Παραγοντικά
Διωνυμικοί Συντελεστές
Γεννήτριες

1. Γενικευμένα Παραγοντικά

$$(\nu)_k = \#\text{ Διατάξεων } \nu \text{ ανά } k = \overbrace{\nu(\nu-1)(\nu-2)\cdots(\nu-k+1)}^{k\text{-παράγοντες}}$$

$\nu \geq k \geq 1$, $\nu, k \in \mathbb{Z}$

Θέλουμε να ορίσουμε
$(x)_k$ για $x \in \mathbb{R}$, $k \in \mathbb{Z}$

▶ Για $x \in \mathbb{R}$, $k \in \mathbb{Z}$, $k \geq 1$ ορίζουμε

$$(x)_k = \underbrace{x(x-1)(x-2)\cdots(x-k+1)}_{k\text{-παράγοντες}}$$

Καθοδικό παραγοντικό k-τάξης του x.

▶ Ιδιότητα:

$$(x)_{k+\lambda} = (x)_k (x-k)_\lambda \qquad k,\lambda \geq 1$$

$$\| $$

$$\underbrace{x(x-1)(x-2)\cdots(x-k+1)}_{k\text{-παράγοντες}}\underbrace{(x-k)(x-k-1)\cdots(x-k-\lambda+1)}_{\lambda\text{-παράγοντες}}$$

▶ Πως πρέπει να ορίσω το $(x)_0$ ώστε η ιδιότητα να ισχύει;

$(x)_{0+\lambda} = (x)_0 (x-0)_\lambda \iff (x)_\lambda = (x)_0 (x)_\lambda$

Άρα ορίζω $(x)_0 = 1$

Επίσης $(x)_{-k} = ;$, $k \in \mathbb{Z}$, $k \geq 1$

Πρέπει $1 = (x)_0 = (x)_{-k+k} = (x)_{-k} (x+k)_k$

Άρα ορίζω $(x)_{-k} = \frac{1}{(x+k)_k}$ για όσα x, k ο παρανομαστής δεν μηδενίζεται

Άρα

$(x)_3 = x(x-1)(x-2)$

$(x)_2 = x(x-1)$

$(x)_1 = x$

$(x)_0 = 1$

$(x)_{-1} = \frac{1}{x+1}$

$(x)_{-2} = \frac{1}{(x+1)(x+2)}$, $(x)_{-3} = \frac{1}{(x+1)(x+2)(x+3)}$

Ομοίως ορίζεται το ανοδικό παραγοντικό k-τάξης του x.

$[x]_k = x(x+1)(x+2)\cdots(x+k-1)$, $x \in \mathbb{R}$, $k \in \mathbb{Z}$, $k \geq 1$

Βασική Ιδιότητα

$[x]_{k+\lambda} = [x]_k [x+k]_\lambda$

$[x]_0 = 1$

$$[x]_{-k} = \frac{1}{[x-k]_k} \qquad x\in\mathbb{R},\ k\in\mathbb{Z},\ k\geq 1$$

$$[x]_3 = x(x+1)(x+2)$$
$$[x]_2 = x(x+1)$$
$$[x]_1 = x$$
$$[x]_0 = 1$$
$$[x]_{-1} = \frac{1}{x-1}$$
$$[x]_{-2} = \frac{1}{(x-1)(x-2)}$$

2. Διωνυμικοί Συντελεστές

$$\binom{x}{k} = \frac{(x)_k}{k!} = \frac{x(x-1)(x-2)\cdots(x-k+1)}{k!} = \begin{bmatrix} x-k+1 \\ k \end{bmatrix} = (-1)^k \begin{bmatrix} -x \\ k \end{bmatrix}$$

$x\in\mathbb{R},\ k\in\mathbb{Z},\ k\geq 0$

$$\begin{bmatrix} x \\ k \end{bmatrix} = \frac{[x]_k}{k!} = \frac{x(x+1)\cdots(x+k-1)}{k!} = \binom{x+k-1}{k} = (-1)^k \binom{-x}{k}$$

3. Σχέση Καθοδικών - Ανοδικών Παραγοντικών

$$(x)_k \overset{\text{α' τρόπος}}{=} x(x-1)(x-2)\cdots(x-k+1) = [x-k+1]_k$$

$k\geq 0$

$$\overset{\text{β' τρόπος}}{=} (-1)^k \, {-x}(-x+1)(-x+2)\cdots(-x+k-1) = (-1)^k[-x]_k$$

$$[x]_k \overset{\text{α' τρόπος}}{=} (x+k-1)_k$$

$$\overset{\text{β' τρόπος}}{=} (-1)^k(-x)_k$$

4 Γεννήτριες Ακολουθιών

$a_0, a_1, a_2, a_3, \ldots$ ⟵ Ακολουθία (a_ν)

$$A(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + \ldots \quad \longleftarrow \text{Γεννήτρια της } (a_\nu)$$

$$= \sum_{k=0}^{\infty} a_k t^k$$

Ακολουθία	Γεννήτριες
(a_ν)	$A(t)$
(b_ν)	$B(t)$
(γ_ν)	$\Gamma(t)$

- $A(t) = B(t) \iff a_\nu = b_\nu \;, \nu = 0, 1, \ldots$
- $A(t) = B(t) + \Gamma(t) \iff a_\nu = b_\nu + \gamma_\nu \;, \nu = 0, 1, \ldots$
- $A(t) = B(t)\Gamma(t) \iff a_\nu = b_0 \gamma_\nu + b_1 \gamma_{\nu-1} + b_2 \gamma_{\nu-2} + \ldots + b_\nu \gamma_0$

$$= \sum_{k=0}^{\nu} b_k \gamma_{\nu-k} \;, \nu = 0, 1, \ldots$$

διότι:

$$a_0 + a_1 \cdot t + a_2 t^2 + \ldots = (b_0 + b_1 t + b_2 t^2 + \ldots)(\gamma_0 + \gamma_1 t + \gamma_2 t^2 + \ldots)$$

$$= \underbrace{b_0 \gamma_0}_{a_0} + \underbrace{(b_1 \gamma_0 + b_0 \gamma_1)}_{a_1} t + \underbrace{(b_2 \gamma_0 + b_1 \gamma_1 + b_0 \gamma_2)}_{a_2} t^2 + \ldots$$

5. Γεννήτρια της $a_k = 1$, $k \geq 0$

$a_k = 1 \quad , k \geq 0$

$$A(t) = \sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} t^k = 1 + t + t^2 + \ldots = \frac{1}{1-t}, \quad |t| < 1$$

6. Γεννήτρια της $a_k = \binom{\nu}{k}$, $k \geq 0$

$a_k = \binom{\nu}{k} \quad k \geq 0$

$$A(t) = \sum_{k=0}^{\infty} \binom{\nu}{k} t^k = \binom{\nu}{0} + \binom{\nu}{1} t^1 + \binom{\nu}{2} t^2 + \ldots + \binom{\nu}{\nu} t^\nu = (1+t)^\nu$$

1η απόδειξη: Επαγωγικά (χρησιμοποιώντας το τρίγωνο Pascal).

2η απόδειξη: Συνδυαστική

$$(1+t)^\nu = \underbrace{(1+t)(1+t)\ldots(1+t)}_{\nu\text{-παράγοντες}}$$

$$= 1 \cdot 1 \cdot \ldots \cdot 1 + 1 \cdot 1 \cdot \ldots \cdot 1 \cdot t + 1 \cdot 1 \cdot \ldots t \cdot 1 + \ldots + t \cdot t \cdot \ldots \cdot t$$

⇒ Άθροισμα 2^ν το πλήθος όρων όπου κάθε όρος είναι της μορφής

$$1^{\nu-k} \cdot t^k \quad ; \quad k = 0, 1, \ldots, \nu$$

Συντελεστής του t^k = # φορών που εμφανίζεται το $1^{\nu-k} t^k$

$\parallel$

μεταθέσεων 2 ειδών στοιχείων

"1" $\rightarrow$ $\nu - k$ φορές

"t" $\rightarrow$ k φορές

$$\frac{(\nu-k+k)!}{(\nu-k)!\,k!} = \frac{\nu!}{(\nu-k)!\,k!} = \binom{\nu}{k}$$

π.χ. $(1+t)^3 = (1+t)(1+t)(1+t)$
$= 1\cdot1\cdot1 + 1\cdot1\cdot t + 1\cdot t\cdot 1 + 1\cdot t\cdot t + t\cdot1\cdot1 + t\cdot1\cdot t + t\cdot t\cdot 1 + t\cdot t\cdot t$

Διωνυμικό Θεώρημα (Θεώρημα Newton)

$$\sum_{k=0}^{\nu} \binom{\nu}{k} \cdot t^k = (1+t)^\nu$$

7. Γεννήτρια της $a_k = \left[{\nu \atop k}\right]$ $k \geq 0$

$a_k = \left[{\nu \atop k}\right]$, $k \geq 0$

$$A(t) = \sum_{k=0}^{\infty} \left[{\nu \atop k}\right] t^k = \left[{\nu \atop 0}\right] + \left[{\nu \atop 1}\right] t + \left[{\nu \atop 2}\right] t^2 + \ldots = (1-t)^{-\nu}$$

Αρνητικό Διωνυμικό Θεώρημα

Απόδειξη

$$(1-t)^{-\nu} = \left(\frac{1}{1-t}\right)^{\nu} = (1+t+t^2+t^3+\ldots)^{\nu}$$

$$= (1+t+t^2+\ldots)(1+t+t^2+\ldots)\ldots(1+t+t^2+\ldots)$$

$\Rightarrow$ Άθροιση όρων της μορφής

$$t^{x_1} \cdot t^{x_2} \cdot \ldots t^{x_\nu} \quad \text{με } x_1, x_2, \ldots, x_\nu \in \mathbb{Z}_{\geq 0}$$

Ο συντελεστής του t^k = # ακέραιων λύσεων της

$$x_1 + x_2 + \ldots + x_\nu = k \quad \text{με } x_i \geq 0$$

$$\left[\begin{matrix} \nu \\ k \end{matrix} \right]$$

8. Βασικές Γεννήτριες

$$\sum_{k=0}^{\infty} t^k = (1-t)^{-1}, \quad |t|<1$$

$$\sum_{k=0}^{\infty} \binom{\nu}{k} t^k = \sum_{k=0}^{\nu} \binom{\nu}{k} t^k = (1+t)^{\nu}$$

$$\sum_{k=0}^{\infty} \left[\begin{matrix} \nu \\ k \end{matrix} \right] t^k = (1-t)^{-\nu}, \quad |t|<1$$

9. Διωνυμικό Πολυωνυμικό Θεώρημα

$$(x_1+x_2)^\nu = \sum_{k=0}^{\nu} \frac{\nu!}{k!\,(\nu-k)!}\, x_1^{k}\, x_2^{\nu-k}$$

$$(x_1+x_2+\cdots+x_r)^\nu = \sum \frac{\nu!}{k_1!\,k_2!\cdots k_r!} \cdot x_1^{k_1}\cdot x_2^{k_2}\cdots x_r^{k_r}$$

$$k_1, k_2, \ldots, k_r \in \mathbb{Z}$$

$$k_1, k_2, \ldots, k_r \geq 0$$

$$k_1, k_2, \ldots, k_r = \nu$$

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ΥΠΟΛΟΓΙΣΜΟΙ ΑΘΡΟΙΣΜΑΤΩΝ

1. Βασικές Γεννήτριες

$$\sum_{k=0}^{\infty} t^k = \frac{1}{1-t}, \quad |t|<1$$

$$\sum_{k=0}^{\infty} \binom{\nu}{k} t^k = \sum_{k=0}^{\nu} \binom{\nu}{k} t^k = (1+t)^\nu$$

$$\sum_{k=0}^{\infty} \left[\nu \atop k \right] t^k = (1-t)^{-\nu}, \quad |t|<1$$

2. Αθροίσματα τύπου $\sum_{k=0}^{\nu} f(k) \binom{\nu}{k} p^k$

π.χ.

$$\sum_{k=0}^{\nu} \binom{\nu}{k} p^k = (1+p)^\nu$$ Διωνυμικό Θεώρημα

$$\sum_{k=0}^{\nu} k \binom{\nu}{k} p^k = ;$$

α' τρόπος (με παραγώγιση)

$$\sum_{k=0}^{\nu} \binom{\nu}{k} t^k = (1+t)^\nu$$

$$\overset{d/dt}{\Longrightarrow} \sum_{k=0}^{\nu} \binom{\nu}{k} k \cdot t^{k-1} = \nu (1+t)^{\nu-1}$$

$$\overset{\cdot t}{\Longrightarrow} \sum_{k=0}^{\nu} \binom{\nu}{k} k \cdot t^{k} = \nu \cdot t (1+t)^{\nu-1}$$

$$\overset{t=p}{\Longrightarrow} \sum_{k=0}^{\nu} \binom{\nu}{k} k p^k = \nu p (1+p)^{\nu-1}$$

Γενικά

$$\sum_{k=0}^{\nu} \binom{\nu}{k} t^k = (1+t)^{\nu}$$

$$\overset{d/dt}{\Longrightarrow} \sum_{k=0}^{\nu} k \binom{\nu}{k} t^{k-1} = \nu (1+t)^{\nu-1}$$

$$\overset{d/dt}{\Longrightarrow} \sum_{k=0}^{\nu} \underbrace{k(k-1)}_{(k)_2} \binom{\nu}{k} t^{k-2} = \underbrace{\nu(\nu-1)}_{(\nu)_2} (1+t)^{\nu-2}$$

$$\overset{d/dt}{\Longrightarrow} \sum_{k=0}^{\nu} \underbrace{k(k-1)(k-2)}_{(k)_3} \binom{\nu}{k} t^{k-3} = \underbrace{\nu(\nu-1)(\nu-2)}_{(\nu)_3} (1+t)^{\nu-3}$$

$$\Longrightarrow \sum_{k=0}^{\nu} \underbrace{k(k+1)\dots(k+r+1)}_{(k)_r} \binom{\nu}{k} t^{k-r} \quad (*)$$

$$= \underbrace{\nu(\nu-1)\dots(\nu-r+1)}_{(\nu)_r} (1+t)^{\nu-r}$$

3. Παράδειγμα

$$S=\sum_{k=0}^{\nu}\underbrace{(3k^2-2k+1)}_{f(k)}\binom{\nu}{k}5^k=;$$

▶Σπάω το πολυώνυμο σε καθοδικά παραγοντικά

$$3k^2-2k+1=\lambda_0(k)_0+\lambda_1(k)_1+\lambda_2(k)_2$$

$$\Rightarrow 3k^2-2k+1=\lambda_0+\lambda_1\cdot k+\lambda_2\cdot k(k-1)$$

$$=\lambda_2 k^2+(\lambda_1-\lambda_2)k+\lambda_0$$

$$\begin{cases}\lambda_2=3\\ \lambda_1-\lambda_2=-2\\ \lambda_0=1\end{cases}\Leftrightarrow\begin{cases}\lambda_2=3\\ \lambda_1=1\\ \lambda_0=1\end{cases}$$

Άρα

$$S=3\sum_{k=0}^{\nu}(k)_2\binom{\nu}{k}5^k+1\sum_{k=0}^{\nu}(k)_1\binom{\nu}{k}5^k+\sum_{k=0}^{\nu}(k)_0\binom{\nu}{k}5^k$$

$$=3\cdot5^2\sum_{k=0}^{\nu}(k)_2\binom{\nu}{k}5^{k-2}+1\cdot5\sum_{k=0}^{\nu}(k)_1\binom{\nu}{k}5^{k-1}+\sum_{k=0}^{\nu}(k)_0\binom{\nu}{k}5^k$$

οπότε

$$S\overset{(*)}{=}75(\nu)_2 6^{\nu-2}+5(\nu)_1 6^{\nu-1}+6^\nu$$

$$=75\nu(\nu-1)6^{\nu-2}+5\nu 6^{\nu-1}+6^\nu$$

4. Εναλλακτικός Τρόπος (με ιδιότητα παραγοντικών)

$$\sum_{k=0}^{\nu} \binom{\nu}{k} t^{k} = (1+t)^{\nu}$$

$$\sum_{k=0}^{\nu} k \binom{\nu}{k} t^{k-1} = \nu (1+t)^{\nu-1}$$

$$\sum_{k=0}^{\nu} k(k-1) \binom{\nu}{k} t^{k-2} = \nu(\nu-1)(1+t)^{\nu-2}$$

Αθροίσματα τέτοιου τύπου υπολογίζονται με βάση:

$$\binom{\nu}{k} = \frac{\nu}{k}\binom{\nu-1}{k-1}, \ k \neq 0$$

$$\frac{\nu!}{k!(\nu-k)!} = \frac{\nu}{k}\frac{(\nu-1)!}{(k-1)!(\nu-k)!}$$

π.χ.

$$\sum_{k=\cancel{0}\,1}^{\nu} k \binom{\nu}{k} t^{k-1} = \sum_{k=1}^{\nu} \cancel{k}\frac{\nu}{\cancel{k}}\binom{\nu-1}{k-1} t^{k-1}$$

$$= \nu \sum_{k=1}^{\nu} \binom{\nu-1}{k-1} t^{k-1 = j}$$

$$= \nu \sum_{j=0}^{\nu-1} \binom{\nu-1}{j} t^{j} = \nu (1+t)^{\nu-1}$$

ή $\displaystyle\sum_{k=\cancel{0}\,2}^{\nu} k(k-1)\binom{\nu}{k} t^{k-2} = \sum_{k=2}^{\nu} \cancel{k}(k-1)\frac{\nu}{\cancel{k}}\frac{\nu-1}{k-1}\binom{\nu-2}{k-2} t^{k-2}$

$$= \nu(\nu-1)\sum_{k=2}^{\nu}\binom{\nu-2}{k-2}t^{k-2} \quad (k-2=j)$$

$$= \nu(\nu-1)\sum_{j=0}^{\nu-2}\binom{\nu-2}{j}t^{j}$$

$$= \nu(\nu-1)(1+t)^{\nu-2}$$

Άρα

$$\sum_{k=0}^{\nu}\underbrace{f(k)}_{\text{πολυώνυμο του } k}\binom{\nu}{k}\rho^{k}$$

1ο βήμα: $f(k) = A_0 + A_1 (k)_1 + A_2 (k)_2 + \dots$

2ο βήμα: $\sum_{k=0} (k)_r \binom{\nu}{k} t^{k-r} = \dots$ Παραγώγιση ή ιδιότητες παραγοντικών

5. Αθροίσματα τύπου $\sum_{k=0}^{\nu}\frac{f(k)}{(k+1)\dots(k+r)}\binom{\nu}{k}\rho^{k}$

π.χ.

$$\sum_{k=0}^{\nu}\frac{1}{k+1}\binom{\nu}{k}3^{k} = ;$$

α' τρόπος

$$\sum_{k=0}^{\nu}\binom{\nu}{k}t^{k} = (1+t)^{\nu} \xRightarrow{\int_0^u dt} \sum_{k=0}^{\nu}\binom{\nu}{k}\int_0^u t^{k}\,dt = \int_0^u (1+t)^{\nu}\,dt$$

$$\Rightarrow \sum_{k=0}^{\nu}\binom{\nu}{k}\frac{u^{k+1}}{k+1} = \frac{(1+u)^{\nu+1}-1}{\nu+1}$$

$$\xRightarrow{\cdot \frac{1}{u}} \sum_{k=0}^{\nu} \frac{1}{k+1}\binom{\nu}{k} u^k = \frac{(1+u)^{\nu+1}-1}{(\nu+1)u}$$

$$\xRightarrow{u=3} \sum_{k=0}^{\nu} \frac{1}{k+1}\binom{\nu}{k} 3^k = \frac{4^{\nu+1}-1}{3(\nu+1)}$$

<u>β' τρόπος</u>

$$\sum_{k=0}^{\nu} \frac{1}{k+1}\binom{\nu}{k} 3^k = \frac{1}{\nu+1}\sum_{k=0}^{\nu} \frac{\nu+1}{k+1}\binom{\nu}{k} 3^k$$

$$= \frac{1}{\nu+1}\sum_{k=0}^{\nu} \binom{\nu+1}{\underbrace{k+1}_{=j}} 3^k = \frac{1}{\nu+1}\sum_{j=1}^{\nu+1} \binom{\nu+1}{j} 3^{j-1}$$

$$= \frac{1}{3(\nu+1)}\sum_{j=1}^{\nu+1} \binom{\nu+1}{j} 3^j = \frac{1}{3(\nu+1)}\left((1+3)^{\nu+1}-1\right)$$

$$= \frac{4^{\nu+1}-1}{3(\nu+1)}$$

6. Παράδειγμα

$$S=\sum_{k=0}^{\nu}\boxed{\frac{k^3-3k^2-2k+5}{(k+1)(k+2)}}\binom{\nu}{k}7^k=;$$

↓

Αθροισμα καθοδικων παραγοντικων

↓

$$A_{-2}(k)_{-2}+A_{-1}(k)_{-1}+A_0(k)_0+A_1(k)_1$$

βαθμός παρανομαστή

διαφορά βαθμων αριθμητή-παρανομαστή

$$\frac{k^3-3k^2-2k+5}{(k+1)(k+2)}=\frac{A_{-2}}{(k+1)(k+2)}+\frac{A_{-1}}{k+1}+A_0+A_1k$$

$$=\frac{A_{-2}+A_{-1}(k+2)+A_0(k+1)(k+2)+A_1(k+1)(k+2)\cdot k}{(k+1)\ (k+2)}$$

$$=\frac{A_{-2}+\boxed{A_{-1}}k+2A_{-1}+A_0k^2+\boxed{3A_0}k+2A_0+A_1k^3+3A_1k^2+\boxed{2A_1}k}{(k+1)\ (k+2)}$$

$$\begin{cases}5=A_{-2}+2A_{-1}+2A_0\\-2=A_{-1}+3A_0+2A_1\\-3=A_0+3A_1\\1=A_1\end{cases}\Longleftrightarrow\begin{cases}A_{-2}=-11\\A_{-1}=14\\A_0=-6\\A_1=1\end{cases}$$

$$\frac{k^3-3k^2-2k+5}{(k+1)(k+2)} = \frac{-11}{(k+1)(k+2)} + \frac{14}{k+1} - 6 + k$$

$$S = -11\sum_{k=0}^{\nu}\frac{1}{(k+1)(k+2)}\binom{\nu}{k}7^k + 14\sum_{k=0}^{\nu}\frac{1}{k+1}\binom{\nu}{k}7^k -$$

$$-6\sum_{k=0}^{\nu}\binom{\nu}{k}7^k + \sum_{k=0}^{\nu}k\binom{\nu}{k}7^k$$

$$= -\frac{11}{(\nu+1)(\nu+2)}\sum_{k=0}^{\nu}\binom{\nu+2}{k+2}7^k + \frac{14}{\nu+1}\sum_{k=0}^{\nu}\binom{\nu+1}{k+1}7^k - 6\cdot 8^\nu +$$

$$+\sum_{k=\cancel{0}\,1}^{\nu}\cancel{k}\,\frac{\nu}{\cancel{k}}\binom{\nu-1}{k-1}7^k$$

$$= -\frac{11}{(\nu+1)(\nu+2)}\frac{1}{7^2}\sum_{j=2}^{\nu+2}\binom{\nu+2}{j}7^j + \frac{14}{\nu+1}\frac{1}{7}\sum_{j=1}^{\nu+1}\binom{\nu+1}{j}7^j -$$

$$-6\cdot 8^\nu + 7\nu\sum_{j=0}^{\nu-1}\binom{\nu-1}{j}7^j$$

$$= -\frac{11}{49(\nu+1)(\nu+2)}\left(8^{\nu+2}-1-7(\nu+2)\right) + \frac{2}{\nu+1}\left(8^{\nu+1}-1\right) - 6\cdot 8^\nu +$$

$$+7\nu\, 8^{\nu-1}$$

7. Σειρές τύπου $\sum_{k=0}^{\infty} f(k)\rho^k$, $|\rho|<1$

π.χ.

$$\sum_{k=0}^{\infty} k\left(\frac{1}{3}\right)^k = 0\left(\frac{1}{3}\right)^0 + 1\left(\frac{1}{3}\right)^1 + 2\left(\frac{1}{3}\right)^2 + \ldots$$

$$\sum_{k=0}^{\infty} t^k = \frac{1}{1-t} \quad |t|<1$$

$$\xRightarrow{d/dt} \sum_{k=0}^{\infty} k t^{k-1} = \frac{1}{(1-t)^2}, \quad |t|<1$$

$$\xRightarrow{\cdot t} \sum_{k=0}^{\infty} k t^k = \frac{t}{(1-t)^2}, \quad |t|<1$$

$$\xRightarrow{t=1/3} \sum_{k=0}^{\infty} k\left(\frac{1}{3}\right)^k = \frac{\frac{1}{3}}{\left(\frac{2}{3}\right)^2} = \frac{3}{4}$$

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Υπολογισμοί Αθροισμάτων

1. Ο γενικός τύπος του Cauchy

$$\binom{r+s}{\nu} = \sum_{k=0}^{\nu} \binom{r}{k}\binom{s}{\nu-k} \quad , \quad r,s \in \mathbb{R}, \ \nu \in \mathbb{Z}, \ \nu \geq 0$$

Απόδειξη

$$(1+t)^x = \sum_{k=0}^{\infty} \binom{x}{k} t^k \ , \ x \in \mathbb{R} \ , \ |t|<1$$

$$\underbrace{\sum_{\nu=0}^{\infty} \binom{r+s}{\nu} t^\nu}_{A(t)} = (1+t)^{r+s} = (1+t)^r (1+t)^s$$

$$= \underbrace{\left(\sum_{k=0}^{\infty} \binom{r}{k} t^k\right)}_{B(t)} \underbrace{\left(\sum_{k=0}^{\infty} \binom{s}{k} t^k\right)}_{\Gamma(t)}$$

$$\Rightarrow a_\nu = b_0 \cdot \gamma_\nu + b_1 \gamma_{\nu-1} + b_2 \gamma_{\nu-2} + \dots + b_\nu \cdot \gamma_0$$

$$= \sum_{k=0}^{\nu} b_k \gamma_{\nu-k}$$

$$\Rightarrow \binom{r+s}{\nu} = \sum_{k=0}^{\nu} \binom{r}{k}\binom{s}{\nu-k}$$

2. Παράδειγμα

$$\binom{\nu}{k} = \frac{\nu}{k}\binom{\nu-1}{k-1}$$

$$\sum_{k=\cancel{0}\,1}^{\nu} k\binom{3}{k}\binom{5}{\nu-k} = \sum_{k=1}^{\nu} \cancel{k}\,\frac{3}{\cancel{k}}\binom{3-1}{k-1}\binom{5}{\nu-k}$$

$$= 3\cdot\sum_{k=1}^{\nu}\binom{2}{k-1}\binom{5}{\nu-k}$$

$$k-1 \to j$$

$$= 3\cdot\sum_{j=0}^{\nu-1}\binom{2}{j}\binom{5}{\nu-1-j} \overset{\text{Cauchy}}{=} 3\cdot\binom{7}{\nu-1}$$

3. Υπενθύμιση

$$\binom{\nu}{k} = \frac{(\nu)_k}{k!} = \genfrac{[}{]}{0pt}{}{\nu-k+1}{k} = (-1)^k\genfrac{[}{]}{0pt}{}{-\nu}{k}$$

$$\genfrac{[}{]}{0pt}{}{\nu}{k} = \frac{[\nu]_k}{k!} = \binom{\nu+k-1}{k} = (-1)^k\binom{-\nu}{k}$$

4. Παράδειγμα

$$\sum_{k=0}^{\nu}\binom{r+k}{k}\binom{s-k}{\nu-k}$$

$$= \sum_{k=0}^{\nu}\binom{(r+1)+k-1}{k}\binom{(s-\nu+1)+(\nu-k)-1}{\nu-k}$$

$$= \sum_{k=0}^{\nu}\genfrac{[}{]}{0pt}{}{r+1}{k}\genfrac{[}{]}{0pt}{}{s-\nu+1}{\nu-k}$$

$$= \sum_{k=0}^{\nu} (-1)^k \binom{-r-1}{k} (-1)^{\nu-k} \binom{-s+\nu-1}{\nu-k}$$

$$= (-1)^{\nu} \sum_{k=0}^{\nu} \binom{-r-1}{k} \binom{-s+\nu-1}{\nu-k}$$

$$= (-1)^{\nu} \binom{-r-1-s+\nu-1}{\nu} = (-1)^{\nu} \binom{-r-s+\nu-2}{\nu}$$

$$= \left[\begin{matrix} r+s-\nu+2 \\ \nu \end{matrix}\right] = \binom{r+s+1}{\nu}$$

5. Αθροίσματα τύπου $\sum_{k=0}^{\nu} \overbrace{f(k)}^{\text{πολυώνυμο}} \binom{r}{k} \binom{s}{\nu-k}$

$f(k) \to$ Άθροισμα καθοδικών παραγοντικών

π.χ.

$$S = \sum_{k=0}^{\nu} (k^2+3k-2) \binom{2}{k} \binom{7}{\nu-k} = ;$$

$$\begin{aligned} f(k) &= k^2+3k-2 \\ &= A_0 + A_1 (k)_1 + A_2 (k)_2 \\ &= A_0 + A_1 k + A_2 k(k-1) \\ &= -2 + 4k + k(k-1) \end{aligned}$$

$$S = -2 \sum_{k=0}^{\nu} \binom{2}{k} \binom{7}{\nu-k} + 4 \cdot \sum_{k=0}^{\nu} k \binom{2}{k} \binom{7}{\nu-k} + \sum_{k=0}^{\nu} k(k-1) \binom{2}{k} \binom{7}{\nu-k}$$

$$S = -2\binom{9}{\nu} + 4\sum_{k=1}^{\nu} \cancel{k}\,\frac{2}{\cancel{k}}\binom{1}{k-1}\binom{7}{\nu-k} + \sum_{k=2}^{\nu} \cancel{k}\cancel{(k-1)}\,\frac{2}{\cancel{k}}\cdot\frac{1}{\cancel{k-1}}\binom{0}{k-2}\binom{7}{\nu-k}$$

$$= -2\binom{9}{\nu} + 8\sum_{j=0}^{\nu-1}\binom{1}{j}\binom{7}{\nu-1-j} + 2\sum_{j=0}^{\nu-2}\binom{0}{j}\binom{7}{\nu-2-j}$$

$$= -2\binom{9}{\nu} + 8\binom{8}{\nu-1} + 2\binom{7}{\nu-2}$$

6. Αθροίσματα τύπου $\sum_{k=0}^{\nu} \frac{f(k)}{(k+1)(k+2)\cdots(k+r)}\binom{r}{k}\binom{s}{\nu-k}$

π.χ.

$$S = \sum_{k=0}^{\nu} \frac{1}{k+1}\binom{r}{k}\binom{s}{\nu-k}$$

$$= \frac{1}{r+1}\sum_{k=0}^{\nu}\frac{r+1}{k+1}\binom{r}{k}\binom{s}{\nu-k}$$

$$= \frac{1}{r+1}\sum_{k=0}^{\nu}\binom{r+1}{k+1}\binom{s}{\nu-k}$$

$$= \frac{1}{r+1}\sum_{j=1}^{\nu+1}\binom{r+1}{j}\binom{s}{\nu+1-j}$$

$$= \frac{1}{r+1}\left(\binom{r+1+s}{\nu+1} - \binom{r+1}{0}\binom{s}{\nu+1}\right)$$

$$= \frac{1}{r+1}\left(\binom{r+s+1}{r+1} - \binom{s}{\nu+1}\right)$$

7. Αθροίσματα σε περιττούς ή άρτιους δείκτες

$$S_a = \sum_{\substack{k=0}}^{\nu} a_k = a_0 + a_2 + a_4 + \dots + \underbrace{a_{2\lfloor \frac{\nu}{2} \rfloor}}_{\text{τελευταίος άρτιος πριν από το } \nu}$$

κ: άρτιος

$$= \sum_{j=0}^{\lfloor \frac{\nu}{2} \rfloor} a_{2j} = j$$

$$S_n = \sum_{k=0}^{\nu} a_k = a_1 + a_3 + a_5 + \dots + a_{2\lfloor \frac{\nu-1}{2} \rfloor + 1} = \sum_{j=0}^{\lfloor \frac{\nu-1}{2} \rfloor} a_{2j+1} = j$$

Βρίσκω το $A(t) = \sum_{k=0}^{\nu} a_k \cdot t^k = a_0 + a_1 t + a_2 t^2 + \dots + a_\nu \cdot t^\nu$

$$\Rightarrow A(1) = a_0 + a_1 + a_2 + \dots + a_\nu$$

$$A(-1) = a_0 - a_1 + a_2 - \dots + (-1)^\nu a_\nu$$

Τότε: $S_a = \dfrac{A(1)+A(-1)}{2}$, $S_n = \dfrac{A(1)-A(-1)}{2}$

8. Παράδειγμα

$$S_a = \frac{1}{1}\binom{\nu}{0} + \frac{1}{3}\binom{\nu}{2} + \frac{1}{5}\binom{\nu}{4} + \dots + \frac{1}{2\lfloor \frac{\nu}{2} \rfloor + 1}\binom{\nu}{2\lfloor \frac{\nu}{2} \rfloor}$$

$$= \sum_{\substack{k=0 \\ \text{κ: άρτιος}}}^{\nu} \frac{1}{k+1}\binom{\nu}{k}$$

Υπολογίζω: $S(t) = \sum_{k=0}^{\nu} \frac{1}{k+1}\binom{\nu}{k} \cdot t^k$

$$= \frac{1}{(\nu+1)t} \sum_{k=0}^{\nu} \binom{\nu+1}{k+1} t^{k+1}$$

$$= \frac{1}{(\nu+1)t} \sum_{j=1}^{\nu+1} \binom{\nu+1}{j} t^j = \frac{(1+t)^{\nu+1} - 1}{(\nu+1)t}$$

$$S(1) = \sum_{k=0}^{\nu} \frac{1}{k+1}\binom{\nu}{k} = \frac{2^{\nu+1} - 1}{\nu+1}$$

$$S(-1) = \sum_{k=0}^{\nu} \frac{1}{k+1}\binom{\nu}{k}(-1)^k = \frac{1}{\nu+1}$$

$$S_a = \frac{S(1)+S(-1)}{2} = \frac{2^\nu}{\nu+1}$$

9 Αθροίσματα με συμμετρία $\binom{\nu}{k} = \binom{\nu}{\nu-k}$

$\sum_{k=0}^{\nu} f(k)\binom{\nu}{k}^2 = \sum_{k=0}^{\nu} f(k)\binom{\nu}{k}\binom{\nu}{\nu-k}$ και μεθοδολογία Cauchy

π.χ.

$$\sum_{k=0}^{\nu} k\binom{\nu}{k}^2 = \sum_{k=\cancel{0}1}^{\nu} k\binom{\nu}{k}\binom{\nu}{\nu-k} = \nu \sum_{k=1}^{\nu}\binom{\nu-1}{k-1}\binom{\nu}{\nu-k}$$

$$= \nu\binom{2\nu-1}{\nu-1}$$

10 Παράδειγμα

$$S=\sum_{k=0}^{\nu}\binom{2\nu}{k}=\binom{2\nu}{0}+\binom{2\nu}{1}+\binom{2\nu}{2}+\dots+\binom{2\nu}{\nu}$$

$$=\binom{2\nu}{2\nu}+\binom{2\nu}{2\nu-1}+\binom{2\nu}{2\nu-2}+\dots+\binom{2\nu}{\nu}$$

$$\Rightarrow 2S=\binom{2\nu}{0}+\binom{2\nu}{1}+\dots+\binom{2\nu}{\nu}+\binom{2\nu}{\nu}+\binom{2\nu}{\nu+1}+\dots+\binom{2\nu}{2\nu}$$

$$=\sum_{k=0}^{2\nu}\binom{2\nu}{k}+\binom{2\nu}{\nu}=2^{2\nu}+\binom{2\nu}{\nu}$$

Άρα $S=\dfrac{2^{2\nu}+\binom{2\nu}{\nu}}{2}$

11. Μη-τυπικά αθροίσματα - Παράδειγμα 1

$$\sum_{s=k}^{\nu}\binom{s}{k}\binom{\nu}{s}\overset{j=s-k}{=}\sum_{j=0}^{\nu-k}\binom{j+k}{k}\binom{\nu}{j+k}$$

$$=\sum_{j=0}^{\nu-k}\frac{(j+k)!}{k!\,j!}\,\frac{\nu!}{(j+k)!\,(\nu-j-k)!}$$

$$=\frac{\nu!}{k!\,(\nu-k)!}\sum_{j=0}^{\nu-k}\frac{(\nu-k)!}{j!\,(\nu-j-k)!}=\binom{\nu}{k}\sum_{j=0}^{\nu-k}\binom{\nu-k}{j}$$

$$=\binom{\nu}{k}2^{\nu-k}$$