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A STRUCTURAL ANALYSIS OF STUDENTS' EPISTEMIC VIEWS

ABSTRACT. Student responses to a structured questionnaire concerned with views on mathematical knowledge, activity and learning, were analysed and interpreted using factorial techniques. The constructs which emerge from the analysis may provide heuristically useful for understanding student beliefs. The findings suggest that there is no simple systematic relationship between beliefs about the nature of mathematical knowledge and activity and about the teaching and learning of mathematics.

BACKGROUND

In recent years, mathematics teaching has been strongly influenced by reforms which emphasise the importance of engaging students more actively in the processes of constructing mathematical knowledge (advocated in the UK, in Cockcroft, 1982; HMI, 1985: in the US, in NCTM, 1980, 1989; NRC, 1989). Although the influences on such reforms have been legion, one significant shaping factor has been concern with what we shall term the *epistemic* aspects of mathematics learning; concerned with students' beliefs about the nature of mathematical knowledge, mathematical activity, and the learning of mathematics.

Although there is now a considerable literature on teachers' epistemic beliefs (reviewed in Thompson, 1992) the same cannot be said for students' beliefs. Drawing inferentially on evidence from classroom activity and discourse, some researchers have developed models of the 'belief systems' of students (Schoenfeld, 1987; Lampert, 1990). Summarising such studies, Schoenfeld (1992, p. 359) includes the following typifications of student beliefs: there is one and only one right answer to any mathematics problem and only one correct method for arriving at it; ordinary students cannot expect to understand mathematics; formal proof is irrelevant to processes of discovery or invention.

Incidental evidence about student perceptions and beliefs can also be found in studies of teaching innovation. For example, in the course of evaluating an exploratory teaching approach, Ruthven (1989) reported on students' perceptions of contrasting teaching styles. These touched on a number of important epistemic issues: the roles of explanation and proof in establishing mathematical knowledge; the place of discussion and argument in mathematical learning; and the relationship between locally-developed and canonical forms of mathematical knowledge.

More recently, Rodd (1993) has provided direct insight into students' epistemic beliefs. She conceptualised beliefs about the nature of mathematics as lying on a dimension defined by 'absolutist' and 'fallibilist' poles, and beliefs about how mathematics is learned as forming a dimension with 'investigative' and 'didactic'

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extremes. Her conjecture was that beliefs on the two dimensions would be strongly associated: the fallibilist pole with the investigative, and the absolutist with the didactic. This, of course, reflects suggestions in the literature on teachers' beliefs: that: "from a Euclidean view, the teacher is the possessor of mathematical knowledge which the pupils must gain. That knowledge is certain, as are the methods used, and teaching becomes a conveying of that knowledge and those methods" (Lerman, 1990, p. 56); "mathematics as a Platonist unified body of knowledge [entails] the teacher as explainer [and] learning as the reception of knowledge; mathematics as problem-solving [entails] the teacher as facilitator [and] learning as the active construction of understanding" (Ernest, 1989, p. 100). However, Rodd's interviews with a small number of students led her to suggest that, for students at least, the relationship between epistemological dispositions and pedagogical preferences is less systematic. We will now present further evidence which supports and clarifies this conclusion.

METHODOLOGY

The data reanalysed here was gathered as part of a wider study of students' proof beliefs, constructs and practices (Coe, 1992). The 70 subjects had followed reformed mathematics curricula over their five years of secondary schooling, and were now nearing the end of the first year of a reformed advanced mathematics course, in a sixth-form college which had pioneered such developments with earlier student cohorts. The students were aged 16 or 17 years; were of above-average attainment for their age cohort, both generally and mathematically; and had chosen to continue to study mathematics at advanced level.

A structured questionnaire was administered to the students. It contained 28 statements about mathematical knowledge, activity and learning, about half of which specifically focused on proof. Respondents were asked to indicate the extent of their agreement or disagreement with these statements on a 5-point scale (agree strongly, scored 2; agree, scored 1; no opinion, scored 0; disagree, scored -1; disagree strongly, scored -2). All but five items produced the full range of responses; and in each of these five cases, only one extreme value was not used.

In view of the possibility of response patterns being influenced by statement sequencing effects (Schuman and Presser, 1981; Dijkstra and van der Zouwen, 1982), the questionnaire was administered in two forms: the second form containing the same items as the first but in reverse order. The two versions were distributed alternately to respondents. In the event, although there was some slight evidence of sequencing effects on individual items, response patterns at the factorial level were robust. This was confirmed by incorporating a variable corresponding to the questionnaire version into the statistical model, and confirming that the resulting factorial pattern was essentially unchanged. Nor was there evidence of any undue acquiescence effect (Schuman and Presser, 1981; Dijkstra and van der Zouwen, 1982; Winkler et al., 1982): 15 of the 28 items produced

TABLE I
Correlations between questionnaire items concerned with the nature of mathematical knowledge

Questionnaire item	Correlations				
Unlike in most other subjects, in maths there is a clear cut right and wrong	•	0.20	−0.05	0.14	0.12
The mathematics developed on another planet would be the same as the mathematics we know		•	0.14	0.02	0.08
If mathematicians today believe a result is true, then mathematicians will still believe it in 1000 years			•	0.02	0.31
"The angles of a triangle add up to a half turn" was true even before any humans recognised it				•	−0.01
Competent mathematicians would always agree about whether or not a proof is valid					•

negative mean scores.

Eight items, only weakly related to the others, were deleted from the analysis. Further assessment confirmed the appropriateness of factorial analysis of the resulting data set[1] and the robustness of the resulting factorial pattern, in particular the appropriateness of an orthogonal axes model[2].

FINDINGS

The first important finding is that students' views were diverse. On 22 of the 28 items, student responses covered the full range, with at least 10% disagreeing and 10% agreeing.

Next, the five items designed, in the original study, to assess students' views of mathematics in terms broadly conceived as absolutist/fallibilist generally produced surprisingly low intercorrelations, both amongst themselves (as shown in Table I) and with other items, and this was reflected in the exploratory analyses (resulting in the exclusion of all but the first tabulated item from the final analysis).

More instructive findings, however, emerged from the factorial analysis of the 20 retained items and the interpretation of the six-factor model of their underlying structure which emerged (as shown in Table II).

The three items with highest loadings on the first factor suggest that the positive pole of this dimension sees proof as a means of generating public and personal confidence in the validity of mathematical relationships. The fourth grouped item asserts the possibility of such an outcome. This factor, then, seems to be concerned with the degree of acceptance of the possibility of achieving certainty in mathematics and the special role of proof in doing so. This dimension is referred to as *proof certitude* (PC).

TABLE II
Factorial structure of cohering questionnaire items
(Only factor loadings accounting for more than 10% of item variance are shown)

Questionnaire item	Factor					
	PC	RP	PR	UF	PI	CA
Once a mathematical result has been proved then you can be certain it is true	0.74					
A proof of a mathematical relationship shows it is true	0.73					
Trying to prove a mathematical relationship for yourself helps to convince you that it is true	0.67					
Unlike in most other subjects, in maths there is a clear cut right and wrong	0.53			0.37		
I do not like to have to explore and investigate in mathematics; I prefer just to know the answers		0.77				
Learning is quicker if you are taught the ideas directly instead of having to find out for yourself		0.64				
I prefer to discover things for myself rather than be told	0.37	−0.57				
My only reason for taking mathematics A-level was to get the qualification		0.52			−0.41	
Trying to prove a mathematical relationship for yourself does not usually help you to understand it better		0.38	0.64			
If a mathematical relationship is obviously true there is no need to justify or prove it			0.63			
There is no point in trying to prove a mathematical relationship unless you are sure it is true			0.59			
The process of trying to prove a mathematical relationship can change your mind about it	0.35		−0.53			
The ability to investigate new situations in mathematics is not as important as knowing mathematical facts				0.69	−0.32	
If a teacher tells me that something is true then I don't need to check it for myself				0.68		

Table II (continued)

Questionnaire item	Factor					
	PC	RP	PR	UF	PI	CA
The ability to investigate new situations in mathematics is not as important as knowing mathematical facts				0.69	−0.32	
If a teacher tells me that something is true then I don't need to check it for myself				0.68		
You can learn better and remember something for longer if you have to discover for yourself				−0.50		
I never believe anything until I can see why it might be true	0.37			−0.43		
Being shown a proof of a mathematical relationship does not help to understand it fully					−0.83	
A proof of a mathematical relationship makes it clear exactly how it depends on other relationships					0.62	
Doing well in maths is more important to me than enjoying the subject						0.85
There is no point in learning anything in maths that is not on the syllabus				0.45		0.58
Percentage of total variance explained	15.5	12.4	9.7	7.4	6.9	6.4

The three items with the highest loadings on the second factor suggest that this dimension expresses the degree of preference for reception over discovery (or investigative) learning. Although the fourth grouped item does not appear to be directly logically related, its association with this dimension is not implausible. This dimension is referred to as *receptive preference* (RP).

Together, these two factors (PC and RP) explain as much variation as the remaining four. They can be taken, then, as defining a plane onto which the most salient aspects of the views expressed by students in response to this set of questionnaire items can be projected. The remaining factors can be seen as elucidating important, but less central, aspects; as helping to capture further detail explaining the variation in students' views, and reflecting their subtlety.

All four items with appreciable loadings on the third factor express more pragmatic reservations about the value of engaging in proof activity: as a means of generating understanding; where a relationship is obvious; where the validity of a relationship is uncertain. This dimension is referred to as *proving reservation* (PR).

Although the items associated with the fourth factor are more disparate, what seems to define the positive pole of this dimension is an emphasis on, and an uncritical acceptance of, a body of authoritatively sanctioned fact. This dimension is referred to as *uncritical factualism* (UF).

Both these factors (PR and UF) load on several items, suggesting that they provide important further insight into the views elicited from students. By comparison, the last two factors (PI and CA) are more restricted in their loadings, suggesting more caution in their interpretation. Equally, however, their more marginal status may simply reflect the original choice of items: in another analysis, they might emerge as more important and central to students' views.

From the two items most highly loaded on the fifth factor, it seems that the positive pole of this dimension is associated with the view that being shown a proof of a relationship provides insight into its cognitive and logical bases. This dimension is referred to as *proof illumination* (PI).

The framework which seems to link the two items appreciably loaded on the last factor is that of success within externally-defined conditions and constraints. Together these items suggest that the positive pole of this dimension expresses commitment to achievement conceived in the relatively narrow terms of examination success. This dimension is referred to as *constrained aspiration* (CA).

Within this structure, it is possible to examine the dispositions of the student group as a whole, by examining the means and standard deviations of individual items within each factor cluster (as shown in Table III).

Although there is clearly a range of views across this group of students on each of these dimensions, it is possible to pick out certain central typifications. There is a relatively strong tendency towards a negative position on *proving reservation* (reflected in the means, all negatively signed and greater in magnitude than 0.6) and less diversity of opinion on this (reflected in the deviations, all less than 1). There is a similar tendency towards a negative position on *uncritical factualism* (all means negatively signed and of magnitude greater than 0.5). On the other factors, the direction of the signed means is not consistent, although there are suggestions of a tendency towards a negative position on *receptive preference* and a positive one on *proof certitude*.

DISCUSSION

From our own professional experience, the constructs which emerge from the preceding analysis and interpretation appear to have considerable heuristic value in understanding students' epistemic beliefs. Equally, however, we would wish to be cautious about the status of these constructs and their structural relations. Essentially, they provide a plausible and parsimonious model summarising the salient features of the data generated by administration of this questionnaire to this group of students. Nonetheless, this evidence does raise some provocative questions.

Particularly interesting is the evidence that these constructs ought to be consid-

TABLE III

Descriptive statistics for cohering questionnaire items (means and standard deviations, items grouped by factor clusters)

Questionnaire item by factor cluster	+/− loading	Mean	Standard deviation
Proof certitude			
Once a mathematical result has been proved then you can be certain it is true	+	0.21	1.19
A proof of a mathematical relationship shows it is true	+	0.39	1.03
Trying to prove a mathematical relationship for yourself helps to convince you that it is true	+	1.16	0.71
Unlike in most other subjects, in maths there is a clear cut right and wrong	+	−0.13	1.19
Receptive preference			
I do not like to have to explore and investigate in mathematics; I prefer just to know the answers	+	−0.50	1.14
Learning is quicker if you are taught the ideas directly instead of having to find out for yourself	+	0.19	1.12
I prefer to discover things for myself rather than be told	−	0.44	0.94
My only reason for taking mathematics A-level was to get the qualification	+	−0.11	1.34
Proving reservation			
Trying to prove a mathematical relationship for yourself does not usually help you to understand it better	+	−0.86	0.89
If a mathematical relationship is obviously true, there is no need to justify or prove it	+	−0.63	0.90
There is no point in trying to prove a mathematical relationship unless you are sure it is true	+	−0.79	0.98
The process of trying to prove a mathematical relationship can change your mind about it	−	1.04	0.58
Uncritical factualism			
The ability to investigate new situations in mathematics is not as important as knowing mathematical facts	+	−0.59	0.97
If a teacher tells me that something is true then I don't need to check it for myself	+	−0.79	0.98
You can learn better and remember something for longer if you have to discover it for yourself	−	1.16	0.85
I never believe anything until I can see why it might be true	−	0.53	1.05
Proof illumination			
Being shown a proof of a mathematical relationship does not help you to understand it fully	−	0.21	1.08
A proof of a mathematical relationship makes it clear exactly how it depends on other relationships	+	0.14	0.79
Constrained aspiration			
Doing well in maths is more important to me than enjoying the subject	+	0.27	1.27
There is no point in learning anything in maths that is not on the syllabus	+	−0.67	1.03

ered as potentially independent dimensions capturing a subtle diversity of student belief. In particular, the independence of *proof certitude, receptive preference* and *uncritical factualism* support the conjecture that there is no simple systematic relationship between students' views of mathematical knowledge and activity on the one hand, and on teaching and learning mathematics on the other. Contrary to what might be suggested, we would not interpret this evidence as illustrating that students are necessarily confused or inconsistent in their beliefs: there may well be plausible constructions and arguments behind a variety of positionings in the 'belief space' that we have defined, and these should be the subject of further investigation.

A distinction also emerges between principled and pragmatic dimensions of belief; best exemplified in the independence of the more principled statements defining *proof certitude* from the more pragmatic ones associated with *proving reservation*. The importance of such a distinction has been confirmed by a separate analysis of the proof constructs and strategies employed by students in carrying out, and accounting for, their coursework (Coe and Ruthven, 1994). Despite the mild positive tendency on *proof certitude* and the stronger negative tendency on *proving reservation* in the students' responses to the questionnaire, the coursework study revealed that, in practice, they were not generally concerned to develop strong proofs of their conjectures. This points to important differences between students' ideal images or espoused beliefs and their practical actions or enacted beliefs. Again, this merits further investigation.

Finally, the evidence presented here, when compared with typifications of student beliefs in more traditional settings, suggests that students who have followed reformed curricula are more diverse in their beliefs, and that some – at least at the level of espoused belief – adopt a more critical perspective towards mathematical knowledge and show a greater appreciation of the role of enquiry in mathematical thinking and learning. Again, this is a conjecture worthy of further exploration.

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NOTES

[1] In particular, the Bartlett coefficient of sphericity was 325.7, the Kaiser-Meyer-Olkin measure of sampling adequancy was 0.57, and 69% of off-diagonal elements of the anti-image correlation matrix were less than 0.09 (SPSS, 1988).

[2] The data set was subjected to factorial analysis with principal components extraction and varimax rotation (SPSS, 1988). These results were compared with those produced by oblimin rotation. Under this alternative, which allows oblique axes, intercorrelations between factors were extremely low, and the factorial structure which emerged was almost identical.

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