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TEACHERS' CONCEPTIONS OF PROOF IN THE CONTEXT OF SECONDARY SCHOOL MATHEMATICS

ABSTRACT. Current reform efforts in the United States are calling for substantial changes in the nature and role of proof in secondary school mathematics – changes designed to provide *all* students with rich opportunities and experiences with proof throughout the *entire* secondary school mathematics curriculum. This study examined 17 experienced secondary school mathematics teachers' conceptions of proof from their perspectives as teachers of school mathematics. The results suggest that implementing "proof for all" may be difficult for teachers; teachers viewed proof as appropriate for the mathematics education of a minority of students. The results further suggest that teachers tended to view proof in a pedagogically limited way, namely, as a topic of study rather than as a tool for communicating and studying mathematics. Implications for mathematics teacher education are discussed in light of these findings.

KEY WORDS: proof, reform, secondary mathematics, teacher conceptions

INTRODUCTION

Many consider proof to be central to the discipline of mathematics and to the practice of mathematicians;[1] yet surprisingly, the role of proof in school mathematics in the United States has been peripheral at best. Proof traditionally has been expected to play a role only in the mathematics education of college-intending students and, even in this capacity, its role has been even further constrained – the only substantial treatment of proof has been limited to the domain of Euclidean geometry. This absence of proof in school mathematics has not gone unnoticed and, in fact, has been a target of criticism. Wu (1996) argued, for example, that the scarcity of proof outside of geometry is a misrepresentation of the nature of proof in mathematics. He stated that this absence is

> a glaring defect in the present-day mathematics education in high school, namely, the fact that outside geometry there are essentially no proofs. Even as anomalies in education go, this is certainly more anomalous than others inasmuch as it presents a totally falsified picture of mathematics itself (p. 228).

Similarly, Schoenfeld (1994) suggested that "proof is not a thing separable from mathematics as it appears to be in our curricula; it is an essential component of doing, communicating, and recording mathematics. And I

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believe it can be embedded in our curricula, at all levels" (p. 76). Sowder and Harel (1998) also argued against limiting students' experiences with proof to geometry, but more from an educational rather than mathematical perspective: "It seems clear that to delay exposure to reason-giving until the secondary-school geometry course and to expect at that point an instant appreciation for the more sophisticated mathematical justifications is an unreasonable expectation" (p. 674).

Reflecting an awareness of such criticism, as well as embracing the important role of proof in mathematical practice, recent reform efforts in the United States are calling for substantial changes in both school mathematics curricula and teachers' instructional practices with respect to proof. In contrast to the status of proof in the previous national standards document (National Council of Teachers of Mathematics [NCTM], 1989), its position has been significantly elevated in the most recent document (NCTM, 2000). Not only has proof been upgraded to an actual standard in this latter document, but it has also received a much more prominent role throughout the *entire* school mathematics curriculum and is expected to be a part of the mathematics education of *all* students. More specifically, the *Principles and Standards for School Mathematics* (NCTM, 2000) recommends that the mathematics education of pre-kindergarten through grade 12 students enable all students "to recognize reasoning and proof as fundamental aspects of mathematics, make and investigate mathematical conjectures, develop and evaluate mathematical arguments and proofs, and select and use various types of reasoning and methods of proof" (p. 56).

Enacting these recommendations, however, places significant demands on school mathematics teachers as approaches designed to enhance the role of proof in the classroom require a tremendous amount of a teacher, particularly in terms of teachers' understanding of the nature and role of proof (Chazan, 1990; Jones, 1997). The challenge of meeting these demands is particularly daunting given that many school mathematics students have found the study of proof difficult (e.g., Balacheff, 1991; Bell, 1976; Chazan, 1993; Coe & Ruthven, 1994; Healy & Hoyles, 2000; Porteous, 1990; Senk, 1985). Further exacerbating these demands is the fact that mathematics teacher education and professional development programs typically have not prepared teachers adequately to enact successfully the lofty expectations set forth in reform documents (Ross, 1998). Consequently, to prepare adequately and support teachers to meet these demands successfully, it is necessary to understand the complex array of factors influencing teachers' interpretations and enactment of such reform recommendations.

One such set of factors, teachers' knowledge and beliefs, have been identified as important determinants of teachers' classroom practices and, consequently, have major implications for the extent to which teachers implement reform recommendations (Borko & Putnam, 1996). Accordingly, the success of current reform efforts with respect to proof depends in large part on the nature of teachers' knowledge and beliefs about proof. Although researchers have focused on teachers' conceptions[2] of proof (e.g., Goetting, 1995; Jones, 1997; Harel & Sowder, 1998; Knuth, In press; Martin & Harel, 1989; Simon & Blume, 1996), this research typically has not focused on teachers as individuals who are teachers of school mathematics; rather, such research has focused primarily on teachers as individuals who are knowledgeable about mathematics. In highlighting the particular focus of this previous research, I am not suggesting that teachers' conceptions as "knowers" of a discipline do not influence their teaching of the discipline. Indeed, in mathematics, for example, there is an extensive body of literature that suggests teachers' subject matter conceptions have a significant impact on their instructional practices (e.g., Fennema & Franke, 1992; Thompson, 1992). Research on teachers' conceptions of proof, however, has tended to focus exclusively on teachers as "knowers" of mathematics rather than as teachers of mathematics. Consequently, research that examines *teachers'* conceptions of proof in the context of secondary school mathematics is greatly needed.

In this article, I describe the results from a study designed both to address this void and to identify areas of need for preparing teachers to enact the recommendations of reform successfully with respect to proof. Prior to presenting and discussing the results of this study, however, I first present a framework for thinking about proof in school mathematics. In addition, I discuss briefly proving practices in school mathematics in terms of this framework.

A FRAMEWORK FOR CONSIDERING PROOF IN SCHOOL MATHEMATICS

Authors have suggested various roles that proof plays in mathematics:

- to verify that a statement is true,
- to explain why a statement is true,
- to communicate mathematical knowledge,
- to discover or create new mathematics, or
- to systematize statements into an axiomatic system (e.g., Bell, 1976; de Villiers, 1999; Hanna, 1983, 1990; Schoenfeld, 1994).

Although these particular roles were proposed in terms of proof in the discipline of mathematics, I have found them to be useful for thinking about proof in school mathematics as well. Accordingly, I have used these five roles as a framework for considering proof in school mathematics in this paper. I elaborate briefly on these roles below.

The role of proof in verifying that a statement is true requires little elaboration. Indeed, few would question that a main role of proof in mathematics is to verify the correctness of a result or truth of a statement (Hanna, 1983). Not surprisingly, this is typically the role most students encounter during their school mathematics experiences. Students' experiences with proof, however, often are limited to verifying the truth of statements that they know have been proven before and, in many cases, are intuitively obvious to them. Such experiences often lead students to view proof as a procedure for confirming what is already known to be true (Schoenfeld, 1994); as a consequence, proof reduces to "just a game because you already know what the result is" (Wheeler, 1990, p. 3).

Mathematicians, however, expect the role of proof to include more than simply the verification of results: "mathematicians routinely distinguish proofs that merely demonstrate from proofs which explain" (Steiner, 1978, p. 135). Making a similar distinction regarding this role of proof, that is, its explanatory role, Hersh (1993) contended that mathematicians are interested in "more than *whether* a conjecture is correct, mathematicians want to know *why* it is correct" (p. 390). Others also have echoed comparable sentiments: "[the] status of a proof will be enhanced if it gives insight as to why the proposition is true as opposed to just confirming that it is true" (Bell, 1976, p. 6) and "the best proof is one which also helps mathematicians understand the meaning of the theorem being proved: to see not only that it is true, but also why it is true" (Hanna, 1995, p. 47). In contrast, proof in school mathematics traditionally has been perceived by students as a formal and, often meaningless, exercise to be done for the teacher (Alibert, 1988). In fact, as Harel and Sowder (1998) suggested, "we impose on them [i.e., students] proof methods and implication rules that in many cases are utterly extraneous to what convinces them" (p. 237). Consequently, as Schoenfeld (1994) concluded, "in most instructional contexts proof has no personal meaning or explanatory power for students" (p. 75).

Many within the mathematics community also view proof as a social construct and product of mathematical discourse (e.g., Davis, 1986; Hanna, 1983; Hersh, 1993; Richards, 1991). As Manin (1977) stated, "a proof becomes a proof after the social act of 'accepting it as a proof'" (p. 48). Similarly, Hanna (1989) noted that "the acceptance of a theorem by practising mathematicians is a social process" (p. 21). Consonant with this view of proof is the approach to mathematical growth and discovery outlined in Lakatos' (1976) seminal book, *Proofs and Refutations*.[3] In addition to the social nature embodied in the process of accepting an argument as a proof, the "product" of such a process (i.e., a proof itself) also provides a means for communicating mathematical knowledge with others (Alibert & Thomas, 1991; Schoenfeld, 1994). Yet, the social nature of proof traditionally has not been reflected in the proving practices of school mathematics. Chazan (1990) suggested that geometry instruction, for example, "downplays any social role in the determination of the validity of a proof; the teacher and the textbook are the arbiters of validity" (p. 20). Balacheff (1991) also noted the limited attention given to the social nature of proof: "What does not appear in the school context is that a mathematical proof is a tool for mathematicians for both establishing the validity of some statement, as well as a tool for communication with other mathematicians" (p. 178).

Proof also plays an important role in the discovery or creation of new mathematics. As de Villiers (1999) noted, "there are numerous examples in the history of mathematics where new results were discovered or invented in a purely deductive manner [e.g., non-Euclidean geometries]" (p. 5). As discussed above, the role of proof in school mathematics typically has been to verify previously known results. The role of proof in creating new mathematics, however, is beginning to play a larger part in many secondary school geometry classrooms, particularly those classrooms in which students are utilizing dynamic geometry software (Chazan & Yerushalmy, 1998). Through their explorations, students generate conjectures and then attempt to verify the truth of the conjectures by producing deductive proofs. In this case, students are using proof as a means of creating new results.

Finally, the role of proof that is the "most characteristically mathematical" (Bell, 1976, p. 24) is its role in the systematization of results into a deductive system of definitions, axioms, and theorems. Although secondary school geometry courses focus typically on a particular axiomatic system (i.e., Euclidean geometry), it is questionable whether students are cognizant of the underlying axiomatic structure. In other words, I surmise (based on my experience both as a former high school teacher and as a teacher educator) that many students view the many theorems that they are asked to prove as essentially independent of one another rather than as related by the underlying axiomatic structure. Geometry instruction typically does not include opportunities for students to reflect on the course from a "meta-level."

In sum, an informed conception of proof in school mathematics, one that reflects the essence of proving in mathematical practice, must include a consideration of proof in each of these roles. There is, however, a long distance between these roles of proof and their manifestation in school mathematics practices (Balacheff, 1991).[4] As a result of such inconsistency, as well as students' inadequate conceptions of proof, current reform efforts are calling for changes in the nature and role of proof in school mathematics (NCTM, 2000; Ross, 1998). My goal in this study was to examine the extent to which teachers are prepared to enact these new recommendations for proof in school mathematics. Specifically, this study examined teachers' conceptions of proof in the context of secondary school mathematics. The study was guided by the following research questions: (1) What constitutes proof in school mathematics? and (2) What are teachers' conceptions about the nature and role of proof in school mathematics?

METHODS

Participants

Seventeen secondary school mathematics teachers (2 middle school and 15 high school teachers) participated in this study.[5] Their years of teaching experience varied from three to twenty years, and the courses they taught varied from 7th grade mathematics to Advanced Placement Calculus. Eleven of the teachers taught either all lower-level mathematics courses (i.e., courses prior to geometry) or mixed-level mathematics courses (e.g., first-year algebra and precalculus), while seven of the teachers taught only higher-level courses (i.e., geometry and above). In addition, the teachers utilized various curricular programs in their classrooms; some of the schools in which the teachers teach have adopted reform-based programs, others utilized more traditional programs. I regarded both course level and curricular program to be possible dimensions of contrast, that is, I hypothesized that teachers may have different conceptions of proof in school mathematics depending on the level of mathematics courses taught or on the curricular program utilized. For example, many reform curricula place an emphasis on open-ended problems for which students are expected to provide justification for their solutions; as a result, teachers may perceive proof as appropriate in courses other than geometry.

The teachers were selected based on their willingness to participate in the study; they were selected from among participants in two ongoing professional development programs. Although one might question how representative the participating teachers are to the larger population of secondary school mathematics teachers, it is worth noting that the participating teachers are committed to reform in mathematics education (as evidenced in their seeking professional development opportunities focusing on reform). Consequently, it is likely that these teachers not only are familiar with the most recent reform documents (e.g., NCTM, 2000) and the corresponding recommendations, but also are interested in changing their instructional practices to reflect more closely the vision of practice set forth in such documents (of which proof is to play a significant role).

Data Collection

The primary sources of data were two semi-structured interviews. Each interview lasted approximately an hour and a half and was audiotaped and later transcribed. The data were collected in two distinct stages, each with a different primary focus. The first stage focused on teachers' conceptions of proof in the discipline of mathematics (i.e., teachers' conceptions as individuals who are knowledgeable about mathematics), while the second stage focused primarily on their conceptions of proof in the context of secondary school mathematics (i.e., teachers' conceptions as individuals who are teachers of secondary school mathematics). At times this separation into two stages seemed somewhat artificial as the teachers often had trouble removing their "teacher hats." Yet, I tried to remain faithful to this separation throughout the data collection stages, often reminding teachers to think about a question or task as someone who is knowledgeable about mathematics rather than as someone who teaches mathematics. Because the focus of this article is on teachers' conceptions of proof in the context of school mathematics, the results presented and subsequent discussion focus primarily on data from the second stage of data collection; however, data from the first stage of data collection are presented as applicable.[6]

The stage two interview questions focused on teachers' conceptions about the nature and role of proof in the context of secondary school mathematics and their expectations of proof for students. Typical questions included: What does the notion of proof mean to you (A question repeated from the first stage)? What constitutes proof in secondary school mathematics? Why teach proof in secondary school mathematics? When should students encounter proof? What do the authors of the NCTM *Principles and Standards for School Mathematics* mean by proof?, and What do you think about the recommendations for proof set forth in the NCTM *Principles and Standards for School Mathematics*?

During the interview, teachers also were presented with different sets of researcher-constructed arguments for several mathematics statements and were then asked to evaluate the arguments in terms of each argument's instructional appropriateness (i.e., would teachers use an argument to convince students of a statement's truth) and provide a rationale for their evaluation. My rationale for including this evaluation as a component of the interview was that I expected the teachers' responses might provide additional insight into their views of proof in school mathematics. The arguments presented were chosen to be appropriate mathematically for secondary school mathematics students. Further, the arguments varied in terms of their validity[7] as proofs as well as the degree to which they were explanatory (see Hanna, 1990, for further elaboration on explanatory proofs). As an example, Figure 1 displays three arguments (from a set of 5) justifying a given statement and which differ in terms of these two variations. The argument presented in (a) is not a proof, while the arguments presented in (b) and (c) are valid proofs. With respect to the arguments' explanatory qualities, argument (a) provides little insight into why the statement is true, while (b) and (c), to varying degrees, do provide insight, that is, they provide "a set of reasons that derive from the phenomenon itself" (Hanna, 1990, p. 9). Although constructing arguments in each set that varied in terms of their explanatory nature required an a priori categorization, I hypothesized that the rationale teachers provided regarding their responses would provide an indication of the degree to which they found particular arguments more or less explanatory than other arguments in a set.

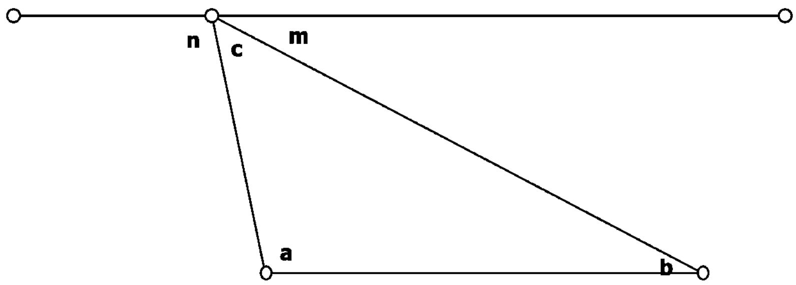
Data Analysis

The data analysis was grounded in an analytical-inductive method in which teacher responses were coded using external and internal codes and then classified according to relevant themes. Coding of the data began using a set of researcher-generated (external) codes that were identified prior to the data collection and that corresponded to, and were derived from, my conceptual framework (i.e., the 5 roles of proof). The deductive approach utilized in producing the external codes was then supplemented with a more inductive approach (Spradley, 1979). As the data were being examined, emerging themes required the proposal of several new codes (e.g., displaying student thinking as a role of proof). After proposing these data-grounded (internal) codes, the data for each individual teacher were then re-examined and re-coded incorporating these new codes. As a means of checking the reliability of the coding and appropriateness of the coding scheme, a second researcher read and coded samples of

(a) I tore up the angles of the obtuse triangle and put them together (as shown below).

The angles came together as a straight line, which is 180°. I also tried it for an acute triangle as well as a right triangle and the same thing happened. Therefore, the sum of the measures of the interior angles of a triangle is equal to 180°.

(b) I drew a line parallel to the base of the triangle.



I know n = a because alternate interior angles between two parallel lines are congruent. For the same reason, I also know that m = b. Since the angle measure of a straight line is 180°, I know n + c + m = 180°. Substituting a for n and b for m, gives a + b + c = 180°. Thus, the sum of the measures of the interior angles of a triangle is equal to 180°.

(c) Using the diagram below, imagine moving BA and CA to the perpendicular positions BA' and CA", thus forming the second figure. In reversing this procedure (i.e., moving BA' to BA), the amount of the right angle, A'BC, that is lost is x. However, this lost amount is gained with angle y (since BA' and DA are parallel, and x and y are alternate interior angles). A similar argument can be made for the other case. Thus, the sum of the measures of the interior angles of any triangle is equal to 180° (Harel & Sowder, 1998).

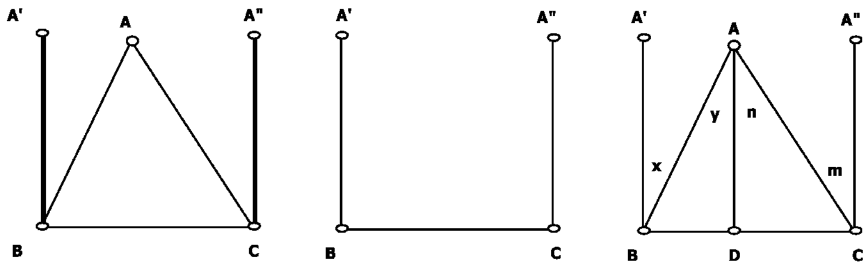


Figure 1. Arguments demonstrating the sum of the angles in any triangle is 180°.

the interview transcripts. The coded samples from both researchers were then compared and differences were discussed until they were resolved. Data were then re-coded taking into account any changes made to the coding scheme. Data for an individual teacher were also examined for consistencies and/or inconsistencies in the nature of their responses; such consistencies/inconsistencies for individual teachers were then examined across data sets for all of the teachers with a focus on themes in the consistencies/inconsistencies noted.

Upon completion of the coding of the data, a domain analysis of the data sets was conducted as a means for identifying, organizing, and understanding the relationships between the primary themes that emerged through the coding process (Spradley, 1979). According to Spradley, domains are categories of meanings that are comprised of smaller categories, the smaller categories being linked to the corresponding domain by a single semantic relationship. Domains selected for this stage of the analysis were informed by the research questions, that is, the issues that were deemed important for this study provided a backdrop against which specific domains were proposed as the data sets were examined. As an example, I used domain analysis techniques to identify the nature of what the teachers seemed to believe constitutes proof in school mathematics – the result was the identification of three levels of proofs (discussed shortly). In this case, the domain chosen was "proof" and the smaller categories, the three levels of proof, were identified as kinds of proof ("kinds of" being the semantic relationship linking the domain to the smaller categories). Similar to the approach taken in coding the data, a more inductive approach supplemented this deductive approach and led to additional domains being proposed.

RESULTS AND DISCUSSION

This section reports and discusses the results of the study and is organized around the two aforementioned research questions: (1) What constitutes proof in school mathematics? and (2) What are teachers' conceptions about the nature and role of proof in school mathematics? Included in the presentation of the results are frequency counts for the relevant themes noted during the data analysis; the counts allow for comparison of the significance of the different themes. In addition, interview excerpts that are representative of particular themes are also provided. Due to space limitations, only themes evident in the responses from at least four teachers are presented (unless a theme is particularly interesting).

What Constitutes Proof in School Mathematics?

In describing what meaning they ascribed to the notion of proof[8] in general (responses were compared from Stages 1 and 2), the majority of the teachers (11) stated, to varying degrees, that a proof is a logical or deductive argument that demonstrates the truth of a premise. The following are representative of the teachers' definitions:

I think it means to show logically that a certain statement or certain conjecture is true using theorems, logic, and going step by step (KK).[9]

I see it as a logical argument that proves the conclusion. You're given a statement, and the logical argument has this statement as its conclusion (SP).

It's the process of justifying a series of steps ... . The justification of the steps is based on already existing mathematical truths (PB).

Other teachers (6) ascribed a slightly more general meaning to proof, that of proof as a convincing argument. For example, one teacher stated that proof is "a convincing argument showing that something that is said to be true is actually true" (KA). Overall, whether defining proof as a deductive argument or as a convincing argument, teachers viewed proof as an argument that conclusively demonstrates the truth of a statement.

Turning now to the meaning of proof in the context of secondary school mathematics, the teachers' descriptions could be categorized using three different degrees of formality: formal proofs, less formal proofs, and informal proofs.[10]

Formal proofs. By many teachers (9), a clear distinction was made between what they considered to be formal proofs and what they considered to be either less formal proofs or informal proofs. The teachers' descriptions of formal proofs were very ritualistic in nature, tied heavily to prescribed formats and/or the use of particular language (cf. Martin & Harel, 1989). For example, one teacher focused on the format required, "You have a prescribed set of rules that you have to follow and a prescribed format" (KA), while another alluded to the particulars of the language used in formal proofs, "Is there a difference between us saying that the angles are equal as opposed to the angles are congruent? ... They need to have before the formal, the informal, where we'll accept either one for now" (FF). Also included in this group of nine teachers were those teachers (4) for whom two-column proofs (i.e., proofs in which statements are written in one column and the corresponding justifications in a second column) are the epitome of formal proofs. "When I think of formal proof, I usually think of the two-column formal proof in geometry"

(NA). Similarly, another teacher commented “When I think of a formal proof, I think of proofs where you have your little ‘T’ [i.e., a spatial description for the organizational structure of a two-column proof]” (DF).

Less formal proofs. Teachers (10) also talked about less formal proofs, proofs which do not necessarily have a rigidly defined structure or are not perceived as being “mathematically rigorous,” but were considered by the teachers to be valid proofs nonetheless (see Figure 1c for an example of a representative argument. These teachers defined less formal proofs more in terms of whether the argument established the truth of its premise for all relevant cases rather than in terms of the rigor involved in the presentation of an argument. Typical definitions included:

Being able to come up with a general statement that always holds (FF).

It’s a convincing argument but it’s generalized . . . . It has to, in some way, be generalized so it’s true for all cases (DL).

It’s a way to decide whether something is true in all situations, or not, based on mathematical justification (PB).

It’s a general argument why something mathematical is true (KB).

In short, the important quality common to arguments of this nature was that they were sound mathematically and proved the general case.

Informal proofs. Finally, all of the teachers considered explanations and empirically-based arguments as representative of informal proofs – arguments not considered to be valid proofs because they are not proofs of the general case (see Figure 1a for an example). Proofs of this nature might best be described as arguments in which one provides reasons to justify one’s mathematical actions or presents examples to support one’s claims (in either case, not arguments one would consider to be valid proofs). In the case of viewing explanations as a type of informal proof, one teacher commented, “They [i.e., students] are always asked to justify their thinking. It seems like proof is everywhere” (SP). In this particular teacher’s case, she is utilizing reform-based curricular materials – materials that frequently ask students to justify the thinking underlying their solutions to presented tasks – thus, her statement that (informal) “proof is everywhere” is not too surprising. In the case of viewing empirically-based arguments as a type of informal proof, another teacher stated, “One of the first [‘proofs’] they do is just prove by a million examples. They can use a bunch of examples and say that’s a proof” (QK).

The Nature of Proof in School Mathematics

As teachers talked about the nature of proof in secondary school mathematics, several themes emerged from the analysis of their responses: the centrality of proof in school mathematics, reform and proof, and students' experiences with proof.

The centrality of proof. In response to being asked if they thought proof should play a central role in secondary school mathematics curricula, teachers expressed varying perspectives depending upon the meaning they associated with proof. The majority of teachers (14) did not consider proof (i.e., formal and less formal proof) to be a central idea throughout secondary school mathematics, questioning its appropriateness for all students. As one teacher stated, "I'm not so sure that we ought to do a lot of teaching of proof" (DL). In general, teacher comments ranged from those that emphasized the types of courses in secondary mathematics for which proof is perceived as appropriate (or inappropriate), for example,

> I think that [i.e., proof] is kind of one of those ivory tower ideas, unless you're teaching in honors pre-calc or honors calc. Actually, that's not true. Any honors class you're going to get into it a little more (FF);
>
> I think if you're asking kids to do them [i.e., proofs], the kids that are going to be able to do them are in the higher level mathematics classes (PB);
>
> In secondary school mathematics proof is not a big part of algebra or analysis [i.e., precalculus] courses (KB);

to those comments that emphasized the type of students who should be provided experiences with proof, such as,

> I think any student going into upper mathematics has to have a strong understanding of proof (KU);
>
> Using 10th grade as a boundary, as opposed to 11th and 12th for kids who are going to be going into mathematics and probably studying mathematics in college, 10th grade and under I'm not convinced that proof has a real role with them (KD).

One teacher even commented that if she were to reduce the amount of material included in secondary school mathematics curricula, proof would be her choice to go. "If you're trying to get through curriculum, then that [i.e., proof] is what I would drop out" (LV). For this latter teacher, proving seemed to be a topic of study rather than a means of coming to understand mathematics.

Thus, for all of these teachers, proof seemed to be an appropriate idea only for those students enrolled in advanced mathematics classes and for those students who will most likely be pursuing mathematics-related majors in college. These views are clearly inconsistent with those represented in current reform documents in which proof is seen as playing a more central role for *all* students: "reasoning and *proof* [italics added] should be a consistent part of students' mathematical experiences in prekindergarten through grade 12" (NCTM, 2000, p. 56). In addition, such views also are inconsistent with the views of mathematics educators who see the importance of proof in a fashion similar to Hanna (1983): "The axiomatic method and the concept of rigorous proof are among the most valuable assets of modern mathematics and should be among the intellectual acquisitions of any high-school student" (p. 4).

In contrast, all of the teachers considered informal proof to be a central idea throughout secondary school mathematics, an idea that was viewed as appropriate for all students and one that should be integrated into every class. One teacher's comment captured this view: "I think informal proof should play a big role. I think we should really work with kids on understanding how the mathematics developed or the justification for it. And pushing them in their work to be able to justify. I think informal proof is really important" (DL). To some extent this view is not surprising; in many respects, the teachers' views are consistent with the messages of earlier reform efforts. The 1989 *Curriculum and Evaluation Standards for School Mathematics* (NCTM) emphasized reasoning and more informal methods of proof as appropriate in the mathematics education of all students – an emphasis that is reflected in the teachers' conceptions of proof in secondary school mathematics. Hanna (1995), although agreeing with the importance of informal proofs in school mathematics, objected to limiting students' experiences with proof to informal methods: "Those who would insist upon the total exclusion of formal methods, however, run the risk of creating a curriculum unreflective of the richness of current mathematical practice. In doing so, they would also deny to teachers and students accepted methods of justification" (p. 46). Similarly, Wu (1996) noted that this emphasis on informal proof, even for students in lower level mathematics classes is "a move in the right direction only if it is a supplement to, rather than a replacement of, the teaching of correct mathematical reasoning; that is, proofs" (p. 226). Yet, as will be discussed shortly, many teachers' conceptions of proof do in fact limit their students' experiences with proof to informal methods.

Reform and proof. Given the disparity in the teachers' views between proof and informal proof as central ideas in secondary school mathematics, the results regarding the teachers' interpretations of the particular recommendations set forth in the *Principles and Standards for School*

Mathematics (NCTM, 2000) with regard to proof are not particularly surprising. All of the teachers expressed one of two opinions, depending on how they interpreted the authors' use of the word *proof* in these recommendations. On the one hand, for those teachers (6) who interpreted the authors' use of proof as mathematical proof (i.e., formal or less formal proof), the recommendations were found to be appropriate only for higher level students (and thus, inappropriate for all students). Typical comments from these teachers included, "I think it depends on the level of the class. Some students are not able to do that [i.e., develop and evaluate proofs]" (NA) and "I think the part about reasoning is okay, but constructing proofs is I think a little much to ask" (DF). One teacher spoke more adamantly about the reality of all students partaking in such recommendations: "I think they're [authors of the *Principles and Standards for School Mathematics*] smoking crack [a drug]. I'd like to see how that would happen, what that looks like in a classroom" (PB).

On the other hand, those teachers (11) who interpreted the authors' use of proof more broadly (i.e., includes formal, less formal, and/or informal proofs), the recommendations were seen as more compatible with their own views regarding the centrality of proof in secondary school mathematics curricula. Several of these teachers differentiated their interpretations of the authors' use of proof by student ability level; as one of these teachers stated:

> Sounds like they want them to be doing proofs throughout 6–12. I think that in itself indicates that they're not expecting rigorous proof in grade 6. They're wanting students to recognize relationships on their own, investigate patterns, use inductive reasoning. At the higher levels learn more rigorous approaches to proving different relationships (KU).[11]

Others interpreted the authors' use of proof as describing different aspects of the entire proving process; thus, "proof" is appropriate for the mathematics experiences of all students. For example, one teacher noted, "They are saying that proof is an integral part of mathematics and it has to deal with reasoning and it has to do with making investigations . . . that's all part of proving" (MQ). Finally, two teachers interpreted the authors' use of proof as being primarily informal in nature. As one of the two teachers commented, "They're still emphasizing, even though they're still using proof a lot in here, it's more the informal way of doing it" (LV). It is apparent from the comments of these eleven teachers that what they consider as mathematical proof (i.e., formal and less formal proofs) is still perceived as appropriate for upper level mathematics students. In effect, the teachers have adopted a pragmatic stance regarding the Standards' use of proof, that is, a particular meaning of proof is utilized depending on the students whom one teaches.

Students' experiences with proof. In response to being asked when students should be introduced to the notion of proof, several teachers (5) suggested that proof in secondary school mathematics is primarily relegated to the domain of Euclidean geometry, and it is in this domain that students actually encounter more formal methods of proof. As one teacher recalled, "I'm not teaching geometry any more, but when I did teach geometry, that [i.e., formal methods of proof] was the focus" (NB). Another teacher provided his rationale for why geometry was the domain of choice: "I like geometry because the medium is a little more concrete. It's a subject matter that you can grasp this whole argument in terms of a formal proof" (CA). Although other teachers did not specifically mention geometry as the "home" of proof in school mathematics, nine additional teachers did view upper level mathematics (including geometry) as appropriate courses for engaging students with proof. Five of these nine teachers, however, viewed proof as being implicit in upper level mathematics courses other than geometry and thus, by default, might be considered to view proof as having an explicit focus only in geometry.

The following is an example of what it means to treat proof as implicit within a non-geometry course. Teachers accepted as valid proofs various algebraic arguments (e.g., a derivation of the quadratic formula), yet, these same teachers stated that if they were to use the given arguments in their own classrooms, they would not discuss them in terms of proof with their students. For the teachers, the proofs would be discussed more as derivations, "I talked about it [i.e., quadratic formula] as a derivation. Here's where the formula comes from" (SP), or as rules, "This is just a rule. We go from here to there" (CC), rather than as proofs in and of themselves. Three of the teachers, however, provided more pragmatic (or pedagogical) reasons for not discussing the arguments as proofs; the following is representative, "I probably would have avoided using proof because ... my experience with the kids is that they would shut down when you use the word proof. They're gone. Shades down" (KD).

Whereas the majority of teachers seemed to view proof as inappropriate for students in lower level classes, all of the teachers reported that they would accept informal proofs (i.e., empirically-based arguments) as proof from their students in lower level mathematics classes. The following teacher's comment is representative: "When they say I noticed this pattern and I tested it out for quite a few cases; you tell them good job. For them, that's a proof. You don't bother them with these general cases" (SP). An unfortunate consequence of such instruction, however, is that students may develop the belief that the verification of several examples constitutes proof (Harel & Sowder, 1998). Wu (1996), recognizing the prevalence of experimentation as a means of establishing truth in secondary school mathematics (and, in particular, in reform-based curricula), warned:

> Now this is not to belittle the importance of experimentation, because experimentation is essential in mathematics. What I am trying to do is point out the folly of educating students to rely solely on experimentation as a way of doing mathematics. Mathematics is concerned with statements that are true, forever and without exceptions, and there is no other way of arriving at such statements except through the construction of proofs (pp. 223–224).

Only two of the teachers, however, heeded Wu's warning and mentioned that they would explicitly discuss the limitations of accepting such arguments as proof with their students. One of these two teachers stated that students need to understand that "demonstrating it [with a few examples] doesn't mean your proof is going to hold true for all cases" (KJ).

In further explaining the role informal arguments play in their classrooms, 11 teachers stated that they would use informal arguments as precursors to the development of more formal arguments in their upper level classes. As one teacher commented: "It's good for students to justify their answers . . . . That's a step into developing proofs, for them to be able to justify their thinking" (MQ). A second teacher described how this process unfolds in her geometry classes: "This [i.e., testing examples to informally prove a statement] students do very early on to show that it works. Then when we introduce other geometry concepts, we come back to this and prove it formally" (SR). Whereas the preceding teacher's comment suggests that her students revisit the initially "proved" statement after acquiring the needed tools to formally prove the statement, another teacher assumed a slightly different perspective regarding the need for generating informal proofs prior to formal proofs. In this case, the teacher viewed the generation of examples as essential to the proving process:

> In order to develop a proof, first off you have to have the insight to say this appears to be happening over here. Why? Looking at what's going on, seeing some interrelationships, there is this idea of using induction and saying it appears that these two, three, or four things are interrelated and they appear to be interrelated in this manner. It appears if I do this, this other thing happens and this is related to this. It could be very simple. It could be very complex in nature. For a proof to really manifest, one needs to have that inductive insight (CA).

Such experiences with more informal methods of proof can provide students with opportunities to formulate and investigate conjectures – both important aspects of mathematical practice – and may help "students develop an inner compulsion to understand *why* a conjecture is true" (Hoyles, 1997, p. 8). Such practices also are reflective of the process of experimentation in mathematics: "Most mathematicians spend a lot of time thinking about and analyzing particular examples. This motivates future development of theory and gives a deeper understanding of existing theory" (Epstein & Levy, 1995, p. 670). Thus, for these teachers, informal proofs were viewed as often serving this very function (in higher level classes), namely, to the "development of theory."

The Role of Proof in School Mathematics

Analysis of the teachers' responses regarding the role of proof in secondary school mathematics revealed several categories – categories reflecting the study's framework (presented at the beginning of the paper) as well as categories that emerged from the data. With respect to the former, the role of proof in systematizing statements into an axiomatic system was the only role associated with the study's framework not mentioned by the teachers.

Developing logical thinking skills. The majority of the teachers (13) identified the development of logical thinking or reasoning skills as a primary role proof plays in secondary school mathematics. Included within this category are teacher responses regarding a role of proof being its applicability to the real world; the applicability role was subsumed by the logical thinking category because the teachers discussed logical thinking skills in terms of the value outside the domain of mathematics as well as inside. Typical comments in response to being asked what role proof serves in secondary school mathematics included the following:

It develops that kind of thinking skill. We naturally use our intuition and we naturally think inductively, but I think getting people to think deductively is not as easy. And that's one thing I think proof causes kids to have to do (KB).

It's not just in mathematics that you use logic. You use it in life problems too . . . . They just can't say just because it is that way. They have to be able to support what they're thnking (KU).

I would really say reasoning skills. Even if you become a carpenter, a businessman, understanding the limitations of your observations and trying to extrapolate them is one thing that I think is really powerful. If you understand your proofs, I think that really builds great reasoning (CA).

Interestingly, one of the teachers, although professing to believe that proof helps develop students' thinking skills, was unsure of its applicability outside of mathematics; "other than the development of reasoning skills . . . I've never had to use proof outside of a math class. I don't know when they might use something like that" (KA).

Communicating mathematics. Ten teachers considered proof in secondary school mathematics to be a social construct. These teachers suggested that in their classrooms, what is accepted as proof is the result of an argument's acceptance as such by the classroom community. One teacher's statement captures this perspective: "They [students' arguments] have to be convincing, accepted by all to be a proof. They [students] may be convinced themselves, but unless they can convince other people, it's not a proof" (EN). Another teacher provided details on how this social process plays out in her classroom: "In class I have kids present their work, then they have a panel of critiquers, and so you can certainly put your work out there for public inspection. The public can do it [i.e., accept an argument as proof]" (SP). A similar process of "public inspection" takes place, reportedly, in another teacher's classroom and serves not only as a means for accepting arguments as proof, but also as a means for making distinctions between proofs and non-proofs.

> I have students come up with different ways of proving something and then discuss which of these really do prove it. They are able to see, able to compare one that does prove it and one that doesn't, and can try to make the distinction between what a proof is and what it's not (KA).

The social nature of proving as described by these teachers, to some extent, reflects the practice of proving in the discipline of mathematics; students present their arguments for public inspection and, as a result of any ensuing deliberations, the arguments are either made more convincing and accurate or are found unacceptable as proofs. Such practices also closely parallel the nature of the practices embodied in visions of reform-based classrooms – classrooms in which students "should expect to explain and justify their conclusions" (NCTM, 2000, p. 342) and in which students "should understand that they have both the right and the responsibility to develop and defend their own arguments" (p. 346).

Displaying thinking. Four teachers indicated that a role of proof in secondary school mathematics was to display students' thinking processes. In other words, a proof provides documentation (oral or written) of how a student arrived at a particular conclusion. Although this role could be perceived as a form of communication, I have chosen to categorize it as a separate role because these teachers seemed to focus more on the display as a means of assessing student understanding. The teachers viewed the display of student thinking as beneficial to the student presenting a proof as well as to the audience reviewing a proof and, in particular, to the teacher assessing the student's level of understanding of the mathematics involved in producing a proof. For example, one teacher commented that presenting a proof allows "students to be able to demonstrate their understanding of why they're able to do something.

I think if they're able to explain a process, their understanding is a bit more solidified" (NA). Another teacher suggested that presenting a proof provides "clarity to their audience or to their teacher that they understand the mathematics they are dealing with" (EN).

Explaining why. The role of proof in explaining why a statement is true surfaced (or failed to surface) in qualitatively different ways. The responses from seven teachers suggested they viewed a role of proof as enabling students to answer why a statement is true. In this case, students learn where statements come from or why they are true rather than accepting their truth as given (from some external source of authority). In this particular category, the focus is not so much on an argument's illumination of the underlying mathematical concepts which determine why a statement is true as much as it is on showing how a statement came to be true. For example, these teachers viewed a proof of the quadratic formula as an illustrative example of the role of proof in answering why something is true. A reader could follow the progression of steps in the derivation to understand *how* the general formula was derived (i.e., "why" it was true). As one teacher commented, "It gives a way [i.e., provides a means] for kids to understand why things are the way they are. Some of the things we say, oh, that's the way things are. Oh, that's the formula. Instead of just accepting at face value, proofs give a way of justifying the formulas" (PB).

Noticeably missing in the teachers' discussions was an explicit recognition of proof serving an explanatory capacity, that is, proof as a means of promoting insight of the underlying mathematical relationships. Five teachers seemed to recognize this construct as evidenced in discussions of their evaluations of the various arguments presented to them. As an illustration of this perspective, one teacher explained why she found the argument presented in Figure 1c to be particularly explanatory:

> It gives you a picture of what's going on and you can see that it's going to be true. You can see how the amount you lose from one of the right angles is made up from the corresponding part of the angle being formed. It is easy to see why the sum is 180° (KA).

None of the teachers, however, explicitly entioned this as a role proof should play in school mathematics. It is possible that the explanatory nature of arguments is not something teachers consciously think about in designing their instruction (cf. Peled & Zaslavsky, 1998). In some respects, it is not surprising that this role was not mentioned by any of the teachers; for many teachers, the focus of their previous experiences with proof as students themselves was primarily on the deductive mechanism or on the end result rather than on the underlying mathematical relationships illuminated by a proof (e.g., Chazan, 1993; Goetting, 1995; Harel & Sowder, 1998). Nevertheless, of all the roles of proof, its role in promoting understanding is, perhaps, the most significant from an educational perspective. In fact, the importance of this role of proof in secondary school is evident in a comment from the Mathematical Association of America's Task Force on the NCTM Standards: "the emphasis on proofs should be more on its educational value than on formal correctness. Time need not be wasted on the technical details of proofs, or even entire proofs, that do not lead to understanding or insight" (Ross, 1998, p. 3). Similarly, Hersh (1993) suggested, "at the high-school or undergraduate level, its [i.e., proof's] primary role is explaining" (p. 398).

Creating mathematics knowledge. Four teachers viewed proof as an opportunity for students to become arbiters of mathematical truth rather than having to rely on their teacher or textbooks to perform this role. In order for students to be autonomous in mathematics classrooms, they must be able to create their own knowledge through validating their own as well as their classmates' knowledge claims. Consequently, this role of proof enables students to become producers of knowledge rather than consumers of other's knowledge. Hanna (1995) saw this role as an inherent characteristic of proof: "Proof conveys to students the message that they can reason for themselves, that they do *not* need to bow down to authority. Thus the use of proof in the classroom is actually *anti*-authoritarian" (p. 46). Accordingly, for these teachers, proof provides students with an opportunity to become mathematically independent thinkers. The following two statements are characteristic of their views of this particular role:

> It allows your students to be independent thinkers, instead of just robots who are told this is the relationship, this is what works, use it to do these problems . . . . Students don't have to rely on a teacher or a book to give them information (KU).

> It's important that they can stand behind a statement or a solution, that they would be able to have a discussion about that, other than saying the teacher told me I was right. That they themselves would have whatever they needed to explain it (EN).

Curricular Program and Course Level Influences

I hypothesized that the nature of the curricular programs used by the teachers and/or the level of the mathematics courses taught might influence teachers' conceptions regarding proof in school mathematics. In the former case, I thought that the conceptions of those teachers who were implementing reform-based curricular programs might be different from the conceptions of those teachers who were implementing traditional curricular programs. Many of the tasks in reform-based curricular materials are open-ended and typically require students to provide justification for their solutions. Such tasks – and the students' corresponding justifications – are very different from the tasks and expected justifications in classrooms using traditional curricular materials (Schoenfeld, 1994). Accordingly, it seemed reasonable to expect that teachers implementing reform-based curricular programs might develop conceptions that differ from the conceptions of teachers implementing traditional curricular programs. With few exceptions, however, the curricular programs from which the teachers taught did not seem to have a significant influence on their conceptions. In other words, of teachers who held the conceptions of proof in school mathematics discussed in the previous sections the number who taught from reform-based curricular programs were relatively equal to the number who taught from traditional curricular programs. One possible explanation (among several possibilities) for the lack of influence might be the treatment of proof in the reform curricular programs, that is, reform curricular programs may place a greater emphasis on informal reasoning than formal reasoning (this conjecture requires further study).

In the latter case, I thought differences in teachers' course loads might influence their conceptions regarding proof in school mathematics. For example, teachers who teach lower-level classes may not see more formal methods of proof as appropriate for their students and, thus, may have developed different conceptions compared to their peers who teach the higher-level classes (classes in which more formal methods of proof might be viewed as appropriate). Once again, however, no significant differences were noted in the conceptions of these two groups of teachers. As an example, virtually all of the teachers agreed that more formal methods of proof were most appropriate in higher-level mathematics courses, while the opposite was true in lower-level mathematics courses – informal proof dominated what the teachers considered to be appropriate experiences for students in these classes.

CONCLUDING REMARKS

As Edwards (1997) suggested, "the teaching of proof that takes place in many secondary level mathematics classrooms has often been inconsistent with both the purpose and practice of proving as carried out by established mathematicians" (p. 187). Consequently, many students do not seem to understand why mathematicians place such a premium on proof (Chazan, 1993). In some sense the foregoing remarks are not surprising;

secondary school mathematics teachers – as well as their students – are, arguably, not mathematicians. Yet, the nature of classroom mathematical practices envisioned by recent mathematics education reform initiatives, and which teachers are expected to establish, reflects the essence of practice in the discipline (Hoyles, 1997). There are examples in the literature of elementary school students engaged in such proving practices (e.g., Ball & Bass, 2000; Maher & Martino, 1996), yet such examples are more the exception rather than the rule, and are rare at the secondary level. At the beginning of this paper, I stated that the purpose of this study was to examine whether secondary school mathematics teachers are prepared to enact in their instructional practices the current reform recommendations regarding proof. The findings of this study suggest that the successful enactment of such practices may be difficult for teachers.

Although teachers tended to view proof as serving several important functions in school mathematics, many of which reflect functions of proof in the discipline, they also tended to view proof as an appropriate goal for the mathematics education of a minority of students. This latter view, however, is clearly inconsistent with the views of those who advocate a more central role for proof throughout school mathematics (e.g., Hanna, 1995; NCTM, 2000; Schoenfeld, 1994). Thus, perhaps the greatest challenge facing secondary school mathematics teachers is changing both their conceptions about the appropriateness of proof for *all* students and their enactment of corresponding proving practices in their classroom instruction. In turn, those parties chiefly responsible for the preservice and inservice education of teachers – mathematics education professionals and university mathematics professors – face the challenge of better preparing and supporting teachers in their efforts to change. A starting point toward helping teachers adopt and implement such a perspective may be to engage teachers in explicit discussions about proof.

In my work with teachers, for example, I have found discussions of questions pertaining to various aspects of proof to be particularly fruitful in getting teachers to reconsider (or at least make explicit) their existing views about proof. Particular questions have included: What is meant by proof? What purpose does proof serve in mathematics? What constitutes proof? Is a proof a proof or are there levels of proof? The latter two questions often have resulted in quite interesting (and, for teachers, often illuminating) discussions. More specifically, these two questions typically arise as teachers engage in the task of evaluating various sets of arguments in terms of their validity as proofs (see Figure 1 for examples of typical arguments). After having debated why particular arguments are or are not valid proofs, many teachers express the realization that their views of what constitutes proof may be too narrowly construed (e.g., proofs require particular format or language).

Relating their responses to the foregoing questions to the context of school mathematics also may serve to refine and extend teachers' views about proof. Other questions have focused specifically on proof in school mathematics and have included the following: Why include proof in school mathematics? Does what suffices as proof in the discipline differ from what suffices as proof in school mathematics? Does what suffices as proof in one course differ from what suffices as proof in another course? What types of experiences with proof should teachers provide students? Further, having teachers construct and present proofs of school mathematics tasks – tasks from various content areas and levels – provides a forum for discussing expectations of proof (e.g., what counts as proof) for students at differing levels of mathematical ability and in different mathematics courses.

I would certainly be remiss not to mention the importance of undergraduate mathematics courses in shaping the conceptions of proof teachers develop. Many of the teachers in this study, for example, viewed proof as a object of study (i.e., a topic one teaches) rather than as an essential tool for studying and communicating mathematics. As Schoenfeld (1994) suggested, "if students grew up in a mathematical culture where discourse, thinking things through, and *convincing* were important parts of the engagement with mathematics, then proofs would be seen as a natural part of their mathematics" (p. 76). Similarly, Harel and Sowder (1998) proposed that "for most university students, including even mathematics majors, university coursework must give conscious and perhaps overt attention to proof understanding, proof production, and proof appreciation as goals of instruction" (p. 275). In turn, such experiences and attention to proof may influence the nature of the experiences with proof that these teachers eventually provide their own students.

The success of teachers in establishing classroom mathematical practices in which proof is an integral part may depend on their changing (or at the very least expanding) their current conceptions of proof in the context of secondary school mathematics. There are certainly conditions other than those discussed in this article which contribute to the formation of teachers' conceptions of proof and the manifestation of these conceptions in their instructional practices, and for which further study is needed. Further, research is needed that examines how teachers' conceptions play out in their day-to-day practices. It is my hope, however, that the findings of this study challenge mathematics educators to recognize and to address the issues related to preparing and supporting secondary mathematics teachers better to successfully enact the newest reform recommendations with respect to proof.

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NOTES

[1] The centrality of proof in mathematics is not without controversy; in fact, the role of proof in mathematics has received increased attention in recent years (e.g., Hanna, 1995; Horgan, 1993; Thurston, 1994). Although such literature contains some interesting discussions among mathematicians about directions related to the future of proof, discussion of this literature is beyond the scope of this paper. The position taken in this article is that proof is, and will continue to be, an important part of mathematical practice.

[2] Rather than treating teachers' knowledge and beliefs as separate domains, I use the term "conceptions" in order to represent the two domains in tandem. While separating teachers' knowledge and beliefs serves as a useful heuristic for thinking about and studying the factors influencing teachers' instructional practices, the separation is less distinct in reality than it is in theory (Grossman, 1990).

[3] Although the social process of proofs and refutations Lakatos described has been criticized for its limited applicability in mathematics (Hanna, 1995), I think the process is worth noting as it undergirds the NCTM (1991) recommendation that teachers establish classroom mathematical practices in which students: "make conjectures and present solutions; explore examples and counterexamples to investigate a conjecture; try to convince themselves and one another of the validity of particular representations, solutions, conjectures, and answers; [and] rely on mathematical evidence and argument to determine validity" (p. 45).

[4] A variety of factors ranging from curricular emphases (e.g., Hoyles, 1997) to psychological issues associated with learning to prove (e.g., Fischbein, 1982) to instructional practices with regard to proof (e.g., Alibert, 1988) have been suggested as contributing to the disparity between proving in mathematical practice and proving in school mathematics. The focus of this paper – teachers' conceptions of proof – is another factor that might be seen to stand between the aforementioned roles of proof and their classroom manifestations.

[5] In the United States, secondary school typically refers to grades 7 to 12 (student ages vary from 12 years to 18 years). For most 7th and 8th grade students, the primary focus of their mathematics courses is on pre-algebra and informal geometry topics (a smaller number of students enroll in a traditional algebra course in 8th grade). At the high school level (grades 9–12), a typical 4-year course sequence is algebra (primarily the study of linear functions), geometry (Euclidean geometry), intermediate algebra (primarily the study of non-linear functions), and pre-calculus (includes the study of trigonometric functions). In some cases, less well prepared students entering 9th grade might enroll in a

pre-algebra course while better prepared students might enroll in a geometry course (thus enabling them to enroll in a calculus course in 12th grade).

[6] See Knuth (In press) for a discussion of teachers' conceptions of proof in the discipline of mathematics.

[7] I recognize that there is not an absolute criterion for the degree of explicitness required in presenting a proof, nor for what mathematical results are acceptable to use; in both cases, the conventions adopted in deciding what counts as a valid proof are those deemed appropriate in a secondary school context (from my perspective as a former secondary school teacher).

[8] Throughout the article, unless the context suggests otherwise, my use of the word proof refers to a deductive argument that shows why a statement is true by utilizing other mathematical results and/or insight into the mathematical structure involved in the statement. When referring to non-proofs, I will use the term argument or informal proof.

[9] KK (a pseudonym) are the initials of teacher who was interviewed.

[10] These categories, formal, less formal, and informal, were researcher-generated based on the teacher responses. It was unclear initially how teachers were using the word *proof*, whether in a mathematical sense or in an colloquial sense; as a result, I introduced the terms as I probed the teachers to further elaborate their usage of the term *proof*. I acknowledge the possibility that by using the term *formal*, I may have influenced teachers to adopt a particular perspective. In many cases, however, teachers actually used the term themselves in making distinctions without my introducing the term.

[11] The data here seem to hold important findings for writers of the NCTM Standards. A common criticism of many reform documents is the level of ambiguity: in this case, the authors never explicitly define what they mean by proof or what proof might look like at various grade levels [the one or two examples that are provided obviously leave teachers unsure how to interpret the recommendations put forth].

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