HOW DO THEORIES INFLUENCE THE RESARCH ON TEACHING AND LEARNING LIMITS OF FUNCTIONS?

Christer Bergsten

Linköpings Universitet

After an introduction on approaches, research frameworks and theories in mathematics education research, theoretical aspects of didactical research on limits of functions are investigated. In particular, three studies with different research frameworks are analysed and compared with respect to their theoretical perspectives. It is shown how a chosen research framework defines the world in which the research lives, pointing to difficulties to compare research results within a common field of study but conducted within different frameworks.

INTRODUCTION

It is generally acknowledged that results from didactical research, as any other research on human behaviour in social settings, depends heavily on the underlying basic assumptions, general approach, and theories and methods used. One may also ask, for a particular study, what factors influence the choice of a specific research framework, and what consequences this choice entails. After a general introduction on research frameworks and the concept of theory, I will go into more detail looking at didactical research on a specific mathematical notion, limits of functions, often referred to as "difficult" for students to learn or understand (Mamona-Downs, 2001). I will give a short overview of some approaches and perspectives used in educational research on limits, and then compare more closely three studies, representing different research frameworks, with a focus on their theoretical underpinnings and claims. In doing this, I will consider the following question: *How does a theoretical basis chosen for a study influence the nature of the purpose, questions, methods, evidence, conclusions, and implications of the study?* This question will be studied using the theoretical notions presented in the next section.

RESEARCH FRAMEWORKS AND THEORIES

In Lester (2005) reasons are given for why educational research needs to be pursued within a scaffolding framework. A framework is here seen as "a basic structure of the ideas (i.e. abstractions and relationships) that serve as the basis for a phenomenon that is to be investigated" (p. 458), representing its relevant features as determined by the adopted research perspective, and serving as a viewpoint to conceptualise and guide the research. A research framework thus "provides a structure for conceptualising and designing research studies", including the nature of research questions and concepts used and how to make sense of data, allowing to "transcend common sense" (p. 458).

According to Eisenhart (1991) three kinds of research frameworks can be identified, that is a *theoretical*, a *practical,* and a *conceptual* framework. Lester (2005) argues that although making the choice of conforming to a particular theory has the advantages of “facilitating communication, encouraging systematic research programs, and demonstrating progress” (p. 459), it also has serious shortcomings, such as prompting explanations by decree rather than evidence, making data “travel” to serve the theory, offering weak links to everyday practice, and limiting validation by triangulation. Also practical frameworks, based on accumulated experiences and ‘what works’, may suffer from limitations caused by norms and narrow insider perspectives. The focus of a conceptual framework is more on justification than on explanation but still based on previous research. Instead of relying on one particular overarching theory as in the case of a theoretical framework, it is “built from an array of current and possibly far-ranging sources”, and can be “based on different theories and various aspects of practitioner knowledge, depending on what the researcher can argue will be relevant and important to address about a research problem” (Lester, 2005, p. 460). The validity for the chosen framework is context dependent, which is its strength considering the implications of the research. Lester thus pragmatically argues with a focus on justification, the purpose of research to answer the *why* questions, that “we should focus our efforts on using smaller, more focused theories and models of teaching, learning and development” (p. 460). The notion of a conceptual research framework relates to the idea of a *networking* strategy for dealing with the diversity of theories within mathematics education (Bikner-Ahsbahs and Prediger, 2006).

Niss (2007) notes that although the notion of theory is essential for mathematics education research, and often used, a definition of the key term *theory* is seldom or never explicitly given. He goes on to offer such a description of this notion, stating that a theory is an organised network of concepts and claims about a domain, where the concepts are linked in a connected hierarchy and claims are either basic hypothesis taken as fundamental, or obtained from these by means of formal or material derivation. To be a theory this network is also required to be stable, coherent, and consistent.

Niss (2007) also separates the purpose of using theory and its role in research. In the former category he lists explanation, prediction, guidance for action or behaviour, a structured set of lenses, a safeguard against unscientific approaches, and protection against attacks from sceptics in other disciplines. Concerning the role of theory he mentions providing an overarching framework, organising observations/ interpretations of related phenomena into a coherent whole, terminology, and research methodology. He also adds that the inclusion of theory in general is needed for publication.

Mathematics education is characterised by its *double nature* (Niss, 1999), with both a descriptive purpose, aimed at increased understanding of the phenomena studied, and a normative purpose, aimed at developing instructional design. In discussing the role of theory in research, the dynamic model presented in Lester (2005) takes this double nature into account (see figure 1). The primary outcome of research may be to increase understanding of a specific phenomenon or to improve practice, a goal pursued along different possible pathways of pure, basic, applied, or developmental research.

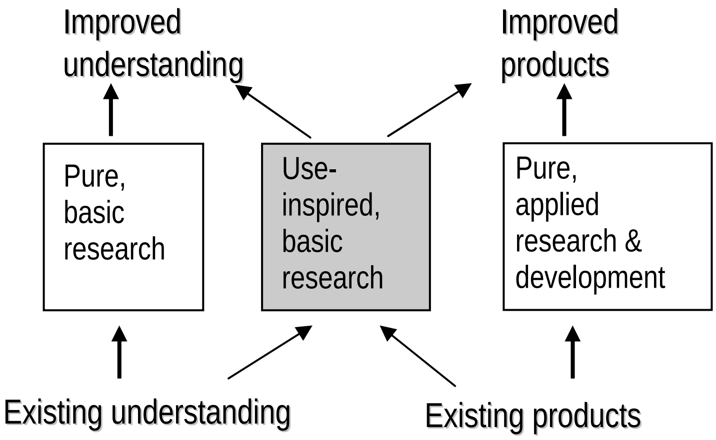

Figure 1. A dynamic model of educational research (Lester, 2005, p. 465)

From a broad perspective, one may identify at least three different general approaches used in research on mathematics education, a cognitive, a social, and an epistemological approach. Within the *cognitive* approach the research interest is focused on the mental structures and thinking processes involved in learning, understanding and doing mathematics, including meta-cognitive dimensions. Taking a classroom perspective, or involving more broad social factors on mathematics education, a *social* approach is used. In an *epistemological* approach, focus is on the structure and use of mathematical knowledge and its diffusion in educational institutions. While acknowledging the fact that, for example, a study with an epistemological approach can use a cognitive as well as a social theoretical framework, or that an epistemological analysis of the object of learning may be used within a cognitive approach, this distinction is made here to identify the *main approach* or focus/interest of the study.

RESEARCH ON THE MATHEMATICAL NOTION OF LIMIT

Overviews of research on limits are found in Cornu (1991) and in Harel and Trgalova (1996). Cognitive approaches have dominated this research, identifying the critical role played by conceptions of infinity, quantification, epistemological obstacles, visualization, concept images, the dialectic between processes and objects, and between intuition and formalism, conceptual metaphors and image schemata, and students' beliefs about mathematics and their role as learners. Epistemological approaches have discussed historical-philosophical aspects of the mathematical ideas involved in the limit concept (Burn, 2005), epistemological obstacles (Cornu, 1991), or contrasted mathematical and didactical organisations observed in classrooms (Barbé *et al*., 2005).

Juter (2006) applies a cognitive approach, using a conceptual framework with a focus on concept images and the "three worlds" of Tall (2004) to investigate Swedish university students' understanding of limits. Her study confirms the image of limits as a problematic area, but that students often tend to overestimate their own abilities as compared to their achievements. Przenioslo (2005) outlines an instructional design based on a "didactical tool" to enable students "to develop conceptions that are closer and closer to the meaning of the concept of limit of a sequence" (p. 90). Mamona-Downs (2001) also aims at developing a teaching/ learning practice by making tacit intuitive views visible and conscious. Bergsten (2006) applies an epistemological approach to analyse university students' work on limit tasks. In the next sections, three studies are described in more detail in order to discuss the consequences of using particular approaches and frameworks. Two of these studies use the same approach but refer to different kinds of research frameworks, while two differ in main approach but are both conducted within a theoretical framework.

APOS theory

An example of a cognitive approach is found in Cottrill *et al*. (1996), where the theoretical framework used is explicitly stated in the paper as the APOS theory, based on Piaget's constructivism. The focus is on students' understanding of the limit concept, and after acknowledging student difficulties to understand this concept, the stated purpose is to "apply our theoretical perspective, our own mathematical knowledge, and our analyses of observations of students studying limits" to develop a "genetic decomposition of how the limit concept can be learned" (p. 167). This tool is based on the APOS theory, in particular how it treats the reconciliation of the dichotomy between "dynamic or process conceptions of limits and static or formal conceptions" (pp. 167-168). The perspective is based on the following statement about mathematical knowledge (p. 171):

> Mathematical knowledge is an individual's tendency to respond, in a social context, to a perceived problem situation by constructing, re-constructing, and organising, in her or his mind, mathematical processes and objects with which to deal with the situation.

The chosen theoretical basis is mirrored in the terminology used, such as the frequent terms *construct* and *schema*, as in "the coordinated process schema is difficult in itself and not every student can construct it immediately" (p. 174). The 'conclusion' is an instructional design focusing on getting students to make "specific mental constructions" (p.169) of importance for understanding the limit concept. The research method is a cyclic process, where a genetic decomposition of the topic is developed by an epistemological analysis. This way the research approach also has a strong epistemological component interacting with the cognitive approach. The genetic decomposition is then forming the basis of an instructional design that is implemented. After extensive observation, and interviews of students, a renewed cycle is performed, which may cause changes in the decomposition and the design, and ultimately also in the theory.

The final genetic decomposition described consists of seven steps (see pp. 177-178), which were materialised in the instructional design. Evidence for students' constructions targeted in the different steps of the decomposition is provided by analyses of interview protocols. Some conclusions about concept development are made, indicating that a "dynamic conception of limit is much more complicated than a process that is captured by the interiorization of an action" (p. 190), and that a strong such conception is needed to move to a formal conception of limit, which is not static "but instead is a very complex schema with important dynamic aspects and requires students to have constructed strong conceptions of quantification" (p. 190).

Reasoning and beliefs

In a study by Alcock and Simpson (2004, 2005), the interaction between students' modes of reasoning (i.e. visual or non-visual) and their beliefs about their own role as learners is investigated. The research is a "naturalistic inquiry into learners' thinking about introductory real analysis" (Alcock and Simpson, 2004, p. 2), with the goal of the study being to "develop a theory of the interactions between various aspects of students' thinking" (p. 7). The approach is thus cognitive and the research framework conceptual, since the study uses theoretical concepts from various sources rather than one overarching theory. Examples of such theoretical concepts used are on visualisation, concept image, spontaneous conceptions (Cornu, 1991), perceptual proof scheme (Harel and Sowder, 1998), semiotic control (Ferrari, 2002), and, for the method, grounded theory, and the distinction account of/account for (Mason, 2002).

The empirical data consist of protocols from interviews with pairs of students, engaged in first-year analysis courses, discussing general issues on university studies, working on given limit problems on sequences and series, and a review of the task session discussing proof and definitions. From the data the observed group of students could be classified either as 'visual' or 'non-visual' depending on their tendencies to introduce diagrams or not during tasks, to use gestures/qualitative terms or algebraic representations when offering explanations, explicitly state their preference or disinclination for pictures or diagrams in reasoning, and to base their sense making to non-algebraic or algebraic reasoning.

The visualizers generally set focus "on the mathematical objects as constructs", draw "quick initial conclusions", and show "Conviction in their own assertions" (Alcock and Simpson, 2004, p. 10). However, a further analysis revealed three "bands" of behaviour of the visualizers, depending on the consistency of the way the mathematical objects were displayed with the formal definitions, and on the ability to use those definitions as a basis for argumentation. These behaviours were found to interact with the students' beliefs about the learner's role. Students that "expect to see consistency and structure" and use "flexible links between visual and formal representations" in mathematics, show an "internal sense of authority", setting value to their own judgement (p. 18). Students using images that are not of sufficient generality to justify their reasoning exhibit a belief that "mathematics will be provided by an external authority" (p. 24). In a similar way, the non-visual students could be divided into three "bands" of behaviour, depending on the accurate use of the mathematical definitions, and on the degree of "semiotic control" connecting algebraic representations with underlying concepts. Also the mathematical behaviour of these students revealed an interaction with their beliefs regarding internal or external authority. The way the course was conducted could not explain the different preferences, and both groups showed a wide range of success and failure, indicating that "there is no "perfect presentation" that will be available to all students" and successful (Alcock and Simpson, 2005, p. 98).

The algebra and the topology of limits

The research presented in Barbé *et al.* (2005) is located in the framework of the Anthropological Theory of Didactics (ATD) and uses the general model of mathematical and didactical activities provided by this theory in terms of mathematical and didactical praxeologies (ibid.). One of the main methodological principles of this research is taking into account how the mathematical knowledge as it is *proposed to be taught* constraints the students' (and the teacher's) mathematical practices. In the case of limits of functions, due to a complex historical process of didactic transposition, the mathematical knowledge *to be taught* appears to be a disconnected union of two mathematical organisations originated by different fundamental questions in the "scholar" mathematical institution: "the algebra of limits" that starts from the supposition of the existence of the limit of a function and poses the problem of how to calculate it for a given family of functions; and "the topology of limits" approaching the question of the nature of the mathematical object "limit of a function" and responding to the problem of the *existence of the limit* of different kinds of functions. Due to traditional tasks and techniques in textbooks and syllabi, the algebra of limits becomes the practical block of the mathematical organisation to be taught, while at the same the theoretical block remains closer to the topology of limits. This mismatch of the two parts of the taught praxeology causes problems for the teacher, as well as the students, to explain, justify, and give meaning to the work on limits. The available theoretical discourse is not appropriate to justify the techniques students learn to use and thus appears to be unmotivated, without any rationale and unable to justify the practice of the algebra of limits – which, for this reason, tends to be considered as a "mechanical" practice difficult to develop. According to the ATD, the main reason for this phenomenon has to be found, not in the human cognition of teachers and students, but in the severe constraints imposed by the process of didactic transposition on the kind of mathematics that can be taught (and learned) at school. Without taking into account these institutional constraints, it seems difficult to understand what teachers and students do (and cannot do) when facing a problem involving limits of functions.

The "split" mathematical praxeology about limits of functions explains some important "distortions" on the teacher's and the students' practice that are due to constraints coming from the first steps of the process of didactic transposition. For instance, the difficulties for the teacher to "give meaning" to the mathematical praxeologies to be taught, because the rationale of limits of functions (why we need to consider and calculate them) cannot be integrated in the mathematical practice that is actually developed at this level. The empirical data for analysing these issues in the particular case reported, were taken from syllabi, textbooks, and classroom observations.

An analysis of influences of theory

An overview of the influence of theory on the three studies discussed above is shown in table 1, structured by the research question stated in the introduction, and by the descriptions, terms and models discussed above in the general section on theory.

The two studies using a cognitive approach both investigate the influence of learning environments on the development of students' understanding of the mathematical concept of limit. The chosen frameworks, however, may be characterized as *closed* and *open*, respectively.

Study	Cottrill *et al.*	Alcock & Simpson	Barbé *et al.*
Main purpose (see figure 1)	Improved understanding and products	Improved understanding	Improved understanding
Research framework	Theoretical: APOS theory	Conceptual: A set of 'local' theories and concepts	Theoretical: ATD
Approach	Cognitive	Cognitive	Epistemological
Questions	How does a 'genetic decomposition' of how the limit concept can be learned look like?	How do various aspects of students' thinking interact?	How are teachers' practices restricted by mathematical and didactical phenomena?
Methods	Research cycle: analysis – design – implementation – observation – analysis	Open and task based interviews	Epistemological analysis and observations of mathematical and didactical organisations
Evidence	Interview protocols	Interview protocols	Syllabi, textbooks, classroom observations
Conclusions	Dynamic conception of limit complicated Formal concept of limit not static Refined genetic de-composition of limit	A theory about the interactions between students' tendency to visualize and beliefs about their own role as learners	The internal dynamic of the didactic process is affected by mathematical and didactical constraints that determine teachers' practice and the mathematics taught
Implications	Further research on quantification needed, along with the genetic decompo-sition, to design effective instruction	At least in small group teaching situations, different students' tendencies to visualize should be taken into account	Problems of motivation, meaning, atomisation of curricula, etc., need a deeper understanding of institutional restrictions regulating teaching

Table 1. The influence of theory on the research process

Cottrill *et al*. (1996) start with, and stay within, a specific theory focusing, along with an epistemological analysis of the limit concept, on the cognitive development of the individual student, forcing interview data to be interpreted in terms of the basic notions of the theory only, that is actions, processes, objects, and schemas: "In trying to fit our observations with the APOS theory, we felt the need to pay more attention to the idea of schema than in our previous work with this theory" (p. 190). The clinical interview is chosen, in line with the Piaget tradition, as the method for collecting evidence on the state of a student's mental schema. This is a closed framework, and the conclusions may be called a progressive confirmation.

As a contrast, the study by Alcock and Simpson (2004) began as "a qualitative investigation of the way different learning environments influence students' developing understanding of real analysis" (p. 1), and the centrality of the distinction between visualizers and non-visualizers, and the interacting role of beliefs, did only emerge by "inductive analysis of the data" (p. 1). This is an indicator of a kind of openness of the conceptual framework chosen. Here the aim was not to develop an instructional design by using a specific theory-based tool, but to increase understanding of the influence of learning environments on students' conceptual understanding. Thus, possibly not to force students' thinking to fit a specific line of development, the data collection method chosen was task solving in pairs, in addition to open questions on general views on mathematics and of proof and definitions. Based on the conceptual framework, which can be seen as emerging from the research problem and the interpretation of data, the conclusion of the research is the development of "a theory which accounts for the students' behaviour" based on the interactions between degrees of visualization and beliefs on authority (Alcock and Simpson, 2004, p. 2).

The study by Barbé et al. (2005) shares with Cottrill *et al*. (1996) a questioning of the mathematical content in use but outlines a very different kind of questioning of this object. While Alcock and Simpson (2004, 2005) take the "scholar" point of view on limits of functions for granted, the theory of didactic transposition allows this questioning. The fact that institutional constraints rarely are taken into account in didactic research makes it difficult to compare results. In Bosch, Chevallard and Gascon (2006) such a comparison between two studies on the concept of continuity is found, focusing on consequences of considering several dimensions of a mathematical practice instead of only one, concluding that "students' difficulties in the learning of a "piece of knowledge" that is praxeologically 'out of meaning' can be taken as a positive symptom of the educational system, instead of a problem in itself" (ibid.).

CONCLUSIONS

The three studies highlighted in this paper all originate from common observations of student 'difficulties' in the mathematical content area of limits of functions, but display, by their different choices of approaches and frameworks, different kinds of research questions and 'answers', based on different kinds of methods and evidence.

The conclusions from the research, in particular, differ considerably at a qualitative level: within the APOS theory, claims are made at a local conceptual and instructional level; within the conceptual framework, a local theory to account for the data is postulated; and within the ATD framework, explanations are found at a systemic level. In addition, the implications listed in table 1 stay for the cognitive approach at a local level of understandings and instruction, while the epistemological approach takes another perspective and considers the level of institutional restrictions as necessary to account for teachers' practice and students' behaviour.

It is evident from these examples how a chosen research framework defines the world in which the research lives, and grows, a fact that also has implications on how to interpret research, and points to the difficult task to compare research results within a common field of study taking into account the different approaches and research frameworks used. This is in itself a research task, and, as a consequence, requires a theoretical stance within which to work. As an example, in this paper specific theoretical tools, based mainly on Lester (2005) and Niss (2007), were chosen as a framework to structure the study of the three studies. But how does this contribute to compare and integrate the contributions of these studies, and others, to a deepened progression of our didactical knowledge of limits of functions?

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