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ORIGINAL ARTICLE

The role of examples in forming and refuting generalizations

Rina Zazkis · Peter Liljedahl · Egan J. Chernoff

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Abstract Acknowledging students' difficulty in generalizing in general and expressing generality in particular, we assert that the choice of examples that learners are exposed to plays a crucial role in developing their ability to generalize. We share with the readers experiences in which examples supported generalization, and elucidate the strategies that worked for us in these circumstances, presuming that similar strategies could be helpful with other students in other settings. We further share several pitfalls and call for caution in avoiding them.

1 Introduction

"Generalization has to do with noticing patterns and properties common to several situations" (Mason, 1999, p. 9). In other words, generalization is afforded by considering particular examples. The choice of these examples is influenced by a variety of factors that depend on the specific context in which the task is set. In this article we first describe specific features of examples that guide learners towards generalization. We focus on two such features: big numbers and numerical variation. We then consider features of counterexamples that help in refuting students' generalizations. Finally, we exemplify potential pitfalls in the choice of examples that may result in wrong generalization. When discussing choices or sets of examples it is appropriate to introduce the notion of *example space* (Watson & Mason, 2005), that is, the pool from which examples are drawn. In this article we explore the connection between example spaces and their role in supporting or impeding generalization.

R. Zazkis (✉) · P. Liljedahl · E. J. Chernoff
Faculty of Education, Simon Fraser University,
8888 University Drive, Burnaby, BC V5A 1S6, Canada
e-mail: zazkis@sfu.ca

2 Background

The importance of generalization in learning has been long acknowledged. Davidov (1972/1990) indicated that "Developing children's generalizations is regarded as one of the principal purposes of school instruction" (p. 10). Focusing on generalization as it pertains to learning mathematics, Lee (1996) suggested that "algebra, and indeed all of mathematics is about generalizing patterns" (p. 103). Further, Mason (1996) claimed: "Generalization is a heartbeat of mathematics. If the teachers are unaware of its presence, and are not in the habit of getting students to work at expressing their own generalizations, then mathematical thinking is not taking place"(p. 65).

However, acknowledging the importance of generalization and its centrality to mathematical experience, it has been widely recognized that mathematical generalization is a challenging task for many learners (e.g. Bills, Ainley & Wilson, 2006; Lee, 1996, Stacey, 1989; Stacey & McGregor, 2001; Becker & Rivera, 2005). This realization posed the following, at times implicit, question to mathematics educators: How is it possible to guide students towards successful generalizations?

In order to address this question, a more refined understanding of what mathematical generalization entails was essential. As such, the efforts of researchers focused on three possible, at times overlapping, directions: (1) classifying different kinds of generalization, (2) classifying different approaches of students towards generalizing, including analysis of their errors, and (3) suggesting different instructional methods of helping students generalize and exploring the efficiency of these methods.

In classifying kinds of generalization as a process researchers distinguished between empirical and structural (Bills and Rowland, 1999) or empirical and theoretical (Dörfler, 1991). Empirical generalization is based on recognizing common features or common qualities of objects. According to Dörfler, empirical generalization entices over-reliance on particular examples and in such is lacking a specific goal to decide what is essential in determining qualities that are relevant for generalization. In fact, Davidov (1972/1990) recognized empirical character of generalizations made by students as one of the sources of difficulties in mastering instructional material. In contrast, in theoretical generalization, essential invariants are identified and substituted for by prototypes. Generalization is then constructed through abstraction of the essential invariants. In a similar fashion, Radford (2003) distinguished between factual, contextual, and symbolic generalization. Factual generalization generalizes numerical action, while contextual generalization also generalizes the objects of these actions. Symbolic generalization involves understanding and utilizing algebraic language. While many researchers criticized empirical generalization, Radford (2003) suggested that factual generalization lays an important foundation to more sophisticated forms of generalization.

In classifying kinds of generalization as an outcome, Harel and Tall (1991) distinguished between (1) *expansive*, where the applicability range of an existing schema is expanded, without reconstructing the schema; (2) *reconstructive*, where the existing schema in reconstructed in order to widen the applicability range; and (3) *disjunctive*, where a new schema is constructed when moving to a new context. Students engaged in disjunctive generalization may construct a separate procedure for a variety of cases and fail to consider earlier examples as special cases of the general procedure. Furthermore, expansive generalization is cognitively easier than reconstructive generalization, but may be insufficient in the long run.

In describing and analyzing students' approaches, Stacey (1989) noted that the majority of 9–13 years old students in her study, when generalizing a linear pattern, used an erroneous direct proportion method, that is, determining the n-th element as the n-th multiple of the difference. Similar results were reported by Zazkis and Liljedahl (2002), where preservice teachers used a "multiple of constant difference" approach when finding a large element in a given arithmetic sequence, ignoring a possible "shifting" of multiples. Orton and Orton (1999) reported the tendency of students to use differences between the consecutive elements in a sequence as their preferred method, and focusing on recursive approach, that frequently prevented them from seeing the general structure of the elements.

Rivera and Becker (2005) classified students' methods in generalizing a pattern as figural, numeric and pragmatic, based on the similarity they recognize in objects. They noted that students who fail in generalizing start by attending to numerical pattern, but do not recognize the connection between different representations. This is consistent with findings of Lannin (2005), who concluded that students who use geometric schemes, that connect the "rule" with visual representation, were more successful in generalizing than students whose scheme was primarily numerical or those using a "guess and check" strategy.

It is natural that having recognized success of some methods, while failure of others, researchers recommend to focus on what brings success. Moreover, as a possible means of encouraging successful generalization, Lannin (2005) suggested that various students' strategies and justifications thereof be brought for scrutiny in front of the classroom for other students to examine their validity and their power. Several researchers suggested the use of spreadsheets as an instructional tool not only for developing generalization but also for expressing it in algebraic terms (Ainley, Bills & Wilson, 2005; Bills, Ainley & Wilson, 2006; Lannin, 2005).

Our study falls within the third direction. We share with readers several strategies that worked for us, presuming that similar strategies could be helpful with other students in other settings. We further share several pitfalls and call for caution in avoiding them. We note the observation of Shiraman (2004), that "problem selection is crucial if the teacher wishes to create problem solving experiences that enable students to generalize" (p. 221). We extend this observation by claiming: the choice of examples is crucial in creating experiences that enable students to generalize.

3 Framework

The framework for our study, both philosophical and methodological, can be best described—using John Mason's notion—as "the discipline of noticing" (Mason, 2002). It is a self-reflection of a group of researchers and teacher educators on some elements of their practice, noticing strategies that worked and identifying a common thread in these strategies.

Following Mason (2006), our "method of enquiry is to identify phenomena [we] wish to study, and to seek examples within [our] own experience" (p. 43). We describe and analyse episodes of instructional uses of examples, with the expectation that the readers may recreate and examine similar experiences in their own practice. In doing so our goal is to "highlight or even awaken sensitivities and awarenesses for them" (Mason, 2006, p. 43) and help readers notice issues that have not been noticed previously.

The philosophical underpinning of this method lies in Mason's (2002) proposal that "Discipline of noticing itself constitutes research method which is particularly suited to practitioners researching their own practice" (p. 183).

Within the discipline of noticing, "data consists of moments of noticing" (Mason, 2002, p. 185). Further, analysis involves examining accounts by, among other methods, interrogating experience, and threading themes. As such, in what follows we share examples of students' successes and failures in generalizing, and of experts' successes and failures in helping students generalize, noticing commonalities in these examples. Some of our examples emerged in our research, others in our teaching practice, others in reflecting upon research of colleagues. They emerged with students of different ages and genders, different backgrounds, different formal experience (from high school students to preservice teachers), and different levels of mathematical sophistication. And they emerged in different settings—from classroom instructions, to conversation with groups of learners, to clinical interviews. Nevertheless, a common theme was noticeable: specific examples that students engage with may support or impede their generalization. As such, our work can be seen as an attempt to generalize about generalizing.

4 Big numbers as a means towards generalization

In this section we explore "big numbers" and "small numbers" in two contexts: algebra and elementary number theory. We first show how the use of "big" numbers can support generalization. We then show how a habitual use of "small" numbers may result in an inappropriate generalization and suggest remediation using "big" numbers. Following these illustrations we address the natural question of "what numbers are big?"

4.1 Algebra: expressing generality in a pattern

Julie was a preservice teacher that participated in a course "Foundations of Mathematics for Teachers", which is a core course for teaching certification at the elementary school level. "Algebra", as a separate topic, was not included in the curriculum for this course; however, basic fluency in generating and manipulating algebraic expressions was expected throughout the whole course. Moreover, a variety of tasks presented to students in this course required the generalization of patterns, where generalizations were expected to be expressed with standard algebraic symbols. Julie, like the majority of students enrolled in this course, did not perceive herself as mathematically inclined. She experienced considerable difficulty in generating and interpreting algebraic notation. In fact, she believed that there was no need to know algebra—which is thought of as a high school topic—for those who are seeking careers in working with "young kids".

In the excerpt of the interview presented below Julie was engaged in the following task:

Consider the following pattern:
$P_1 = 2 \times 3$
$P_2 = 3 \times 4$
$P_3 = 4 \times 5$ and so on.
What is P_{100}? What is P_n?

Julie's first attempt at the task was to express the first numbers of the pattern as 6, 12, 20, 30, 42..., and to note the increasing sequence of differences. However, as evident in the interview excerpt below, when requested to consider P_{100} she seemed to note the explicit pattern of factors.

Interviewer: Can you write an expression for P_n?
Julie: I'm not sure what P_n is.
Interviewer: It's just a way to refer to the n-th number in our pattern. See, we called our first number P_1, the second P_2 and so on, so we shall call 75th number P_{75}, n can be any number...
Julie: So the answer will depend on the number.
Interviewer: Can you write down an expression, that depends on n, that will tell us what P_n is?
Julie: I'm not sure how
Interviewer: OK. Suppose $n = 100$. What is P_n if n is 100
Julie: It's 10,100. Sorry, wait, [using a calculator] it's 10,302, yea, 10,302.
Interviewer: How did you get it?
Julie: 101 × 102 is 10302, I first did 100 × 101, but I should go one further.
Interviewer: What if n is 173?
Julie: So it will be 173 times, no, 174 times 175, will be 30,450.
Interviewer: Could you please write an expression for P_{173}, not the number itself, but the way to get it
Julie: [writes 174 × 175]
Interviewer: So what about n, can you write an expression for n?
Julie: It's the next one times the next one, but you give me n, then I will do the timesing.
Interviewer: OK, n is 3^{100}.

Julie: [tries to plug numbers into calculator]. I don't know what it is.
Interviewer: It is 3^{100}. You don't have to calculate it. Just write an expression using this number.
Julie: You need the next one.
Interviewer: Right. You need the next one. What will it be?
Julie: [hesitates] $3^{100} + 1$?
Interviewer: So..?
Julie: [writes $3^{100} + 1 \times 3^{100} + 2$], Is this what you want?
Interviewer: Does this tell you how to find the number?
Julie: Sort of, it doesn't tell me what the number is.
Interviewer: It's fine. All we need to know here is, if your number is unknown, let's call it n, how would you get P_n
Julie: n + 1 times n + 2 you mean? Like this? (writes down the expression, forgetting the parentheses)

Julie definitely recognized the pattern, but had difficulty expressing generality in algebraic terms. She immediately calculated P_{173} as 174×175, however, she saw 174 as "the next one" rather than 173 + 1. For many students accustomed to algebra, the expression $(n + 1)(n + 2)$ appears easier to grasp than $(3^{100} + 1)$ $(3^{100} + 2)$. However, for those experiencing difficulties with expressing generality in algebraic terms, consideration of a specific big number can prove helpful.

It has been noted that even students who are comfortable working with specific cases, have difficulty in expressing generality. Following a teaching experience focusing on algebraic generalization of patterns, Lee (1996) observed that the "major problem was not in *seeing a pattern*, it was in perceiving an algebraically useful pattern" (p. 95). Like Lee's students, Julie has no difficulty in "seeing the pattern". Her difficulty is with expressing the pattern in terms that can lead to generalization. Further, Zazkis and Liljedahl (2002) noted that there is a significant gap between recognizing a pattern and being able to express it algebraically. In their research students had little difficulty describing a pattern verbally and making a prediction based on the identified relationships in a pattern, but were not able to provide a formal algebraic description.

Zazkis and Gadowsky (2001) suggested that many students consider only the decimal representation of a number to be "a number." Other forms of number representation, such as prime decomposition or sum of numbers, were considered as "expressions" or "exercises," rather than numbers. Julie in the above excerpt experienced a similar difficulty. She wanted her result to be a number, not a numerical or algebraic expression. However, achieving a numerical expression that was not to be computed assisted Julie towards generating an algebraic expression. It demonstrates how a pedagogical strategy of considering "big" numbers may serve as a stepping stone towards expressing generality with algebraic symbols.

4.2 Algebra: attending to generality in short multiplication

"Big" numbers can be helpful in drawing students' attention to underlying structure in algebraic expressions. Consider for example the difference of squares in

$$(2x - 3y)^2 - (x + 3y)^2.$$

When a request to simplify this expression was presented to high school students, there was an almost instinctive desire to remove the parentheses by first computing the squares of the sum and the difference. Indeed, when numbers are small, there is no apparent advantage in explicitly attending to the structure of difference of squares. However, students generated much more appreciation for this general structure when the chosen numbers were not quite "big", but just bigger, such as

$$(26x - 15y)^2 - (24x + 15y)^2$$

or

$$(255a - 15b)^2 - (245a + 15b)^2.$$

Those who attended to the difference of squares were able to simplify the expressions significantly faster and without a calculator. This encouraged other classmates to employ this approach as well.

4.3 Prime numbers: generalizing from prior experience

There are infinitely many primes. Consequently, there are many more "large primes," than "small primes," regardless of how "large" and "small" are defined. However, in school students seldom see an example of a prime number which is larger than 100. In fact, most of the examples are limited to primes below 31. Exercises involving prime decomposition usually result in finding "small" prime factors such as 3, 5 or 7. Exposure to these, and only these, examples leads to at least two incorrect generalizations about primes: prime numbers are small and every composite number should have a small prime factor. These beliefs are often not mentioned explicitly, but they prevail in students' responses to a variety of tasks. Further, these beliefs seem to co-exist with participants' awareness of the existence of infinitely many primes as well as the existence of very large prime numbers. Prior research documented a variety of excerpts that illustrate these incorrect generalizations (Zazkis & Campbell, 1996; Zazkis & Liljedahl, 2004).

As an example, we consider here an excerpt from the clinical interview with Tanya, a preservice elementary school teacher. She responded to the question of whether 391 was divisible by 23 by dividing 391 by a few "small" primes and claiming that 391 was prime (note that $391 = 17 \times 23$) (Zazkis & Campbell, 1996).

> Tanya: I don't know. I guess, like I, um, like I was saying with, I know there's a way to do it, prime factorization, and I know that 23 is a prime number, but I guess, um, I was assuming, for some reason, that as long as 391 was not a prime number, it would have a factor smaller than 23, a prime factor smaller than 23.
> Interviewer: And is there a reason why, why you thought that way?
> Tanya: Um, I guess because in, in my experience in most cases, a large number, relatively large number like 391, would have, well any number not even a large number, any number has um some small prime factors in addition to whatever else we have, we may have a large number, prime factors like 23, but they also tend to have things like 2 and 3 and 5 and 7.
> Interviewer: Well what if we took 2 very large prime numbers?
> Tanya: Um hm...
> Interviewer: And multiplied them together to get another number?
> Tanya: Um hm.
> Interviewer: Would that number have a small prime in its prime factorization?
> Tanya: (Pause) Umm, no, I don't think so.

[...]

> Tanya: I guess it's probably just more experience than anything, but it just seems to me that when you factor a number into its primes, I mean what you're doing is, you're trying to find the smallest, I mean numbers that can no longer be broken into anything smaller aside from 1 and itself, so that, I guess it's just the whole idea of factoring things down into their smallest parts...
> Interviewer: Um hm.
> Tanya: I guess gives me the idea that those parts are themselves going to be small.

As stated earlier, similar inappropriate generalizations with respect to "small primes" were worded explicitly only occasionally, as in Tanya's case, but could be derived from students' actions and approaches to problem solving.

According to Tall and Vinner (1981) an individual's concept image is not constant, it may grow and change with experience and its various parts develop at different times and in different ways. We suggest that specific examples of the concept to which students are exposed are part of such experience. If we agree with the claim that the students' concept image is influenced by examples, then a reasonable approach to reconstructing their image is to create a richer set of examples, that is, to extend the example space from which generalizations are drawn. The availability of calculators makes this approach possible. We advocate calculator supported activities in checking for primality of large numbers and decomposing composite numbers into "large" primes. Unfortunately, the majority of textbook exercises and examples are still focusing on single digit prime numbers and as such reinforcing the mis-generalization of primes being "small" rather than contributing to its reconstruction.

4.4 What is a "big" number?

Of course, a natural question arises: what number is big? What prime is large? While there is no deterministic answer—but rather contextual and situational—an analysis of our previous examples suggests a working definition. Our initial example considers a number to be "big" if it is beyond the computational abilities of a hand-held calculator. In our second example numbers are "big" simply if they do not invite immediate computation and direct learners to reconsider structure more carefully. In our third example a prime is "large" if it is neither 2, 3, 5 or 7 nor if it is instantly recognized as prime, that is, it does not belong to the repertoire of "primes" immediately retracted from one's memory.

An underlying theme that is associated with the different uses of "big" numbers is numerical variation. Big numbers were chosen to accentuate the invariant structure; however, numerical variation is not restricted solely to the use of big numbers, as we describe in the next section.

5 Numerical variation as a means towards recognizing general structures

In this section we show how numerical variation—that is, changing numbers in the tasks while keeping the structure invariant—is a helpful strategy on a pathway to a solution. To exemplify numerical variation as a means towards generality we consider two classic puzzles and two rather conventional, but troublesome problems.

5.1 On chickens, eggs and grains

Consider the following well-known riddle:

> If a hen-and-a-half lays an egg-and-a-half in a day-and-a-half, how many days does it take one hen to lay one egg?

Many students either answer "one day" by inertia or claim that the problem presents impossible nonsense. Only few suppress these tendencies and attempt to reason through the available information. What does this twist on a chicken and an egg problem have to do with generalization? We believe that by the end of this section the connection will become clear. However, let us consider first a more "realistic" problem.

> A pound of fancy grain cost \$1.68, how much grain can you buy for \$0.50?

We presented this problem to various populations, from middle school students to preservice elementary school teachers, and there is a significant number of people who were making errors in setting up the division statement, that is, dividing 1.68 by 0.50 rather than 0.50 by 1.68. What is the best way to help them? Of course, pointing to their error is not helpful beyond the given problem.

The general multiplicative structure that a learner needs to recognize in order to solve this problem is a pound of fancy grain costs X, how much grain can you buy for Y? This is an example of a more general form of quotative (measurement) division, that is, division structure that determines how many times can X fit into Y, or how Y can be measured with X.

Once the structure is recognized, the solution is given by $Y \div X$. The question, however, is what is it that can guide learners towards seeing the generality in this particular case (Mason & Pimm, 1984)? What we found helpful is changing the numbers.

> A pound of fancy grain cost \$2, how much grain can you buy for \$6?
> A pound of fancy grain cost \$2, how much grain can you buy for \$20?

The numbers in these examples are compatible, that is, easily manipulated and work well together. Learners seldom have problems with these kinds of questions, so using them as a starting point is beneficial. Once the general structure is established, it is possible to move to "more problematic" numbers, involving fractions.

> A pound of fancy grain cost \$2, how much grain can you buy for \$0.50?

And then gradually return to the original problem.

This strategy can be seen as a modification of the "structured variation grids" (Mason, 2001, 2007) in that it is a gradual numerical variation for the purpose of prompting recognition of structure. So, why is the structure more readily recognized when numbers are compatible than when they are not? We suggest that the source of the obstacle is with the perceived range of permissible change. That is, the numbers in the initial problem are "too far" from the students' example space of problems that are associated, implicitly, with measurement division. Numerical variation assists in recognizing similarities and extending the general structure, a step necessary for the solution.

Now we return to chickens and eggs.

> If six hens lay six eggs in 1 day, how long will it take one hen to lay one egg?

This sounds close to trivial. We can keep "one chicken" invariant and ask further:

> If six hens lay six eggs in a day and a half, how long will it take one hen to lay one egg?

Or

> If six hens lay six eggs in 6 days, how long will it take one hen to lay one egg?

This apparent analogy to the initial problem suggests a solution.

5.2 On "big" percentages

We often smile when someone claims to be putting 120% of his energy in a project or being 200% sure of something. These claims exemplify a tendency to overemphasize an effort or certainty, rather than provide an accurate measure. When a whole is 100%, what is indicated by a percentage higher than 100? We found that when a high percentage appears in a mathematical problem situation it often leads the learners away from recognizing the general structure. Consider for example the following problem:

> The price of a can of coffee was \$10. It increased by 400%, what is the new price?

In a class of preservice elementary school teachers, about half of the students claimed that the new price was \$40, explaining that 400% meant "quadrupling." Once again, what we found helpful towards recognizing the general strategy is numerical variation:

> The price of a can of coffee was \$10. It increased by 20%, what is the new price?
> The price of a can of coffee was \$10. It increased by 35%, what is the new price?

The price of a can of coffee was $10. It increased by 100%, what is the price now?

Again, we believe that the main problem is with the perceived range of permissible change. While 20, 35%, or even 100% fits within what is expected—both in a real world context and in a mathematics classroom context—the increase of 400% appears beyond a "reasonable" permissible change. We now turn to another popular riddle, and attempt to explain it with numerical variation.

5.3 On bellboy and gentlemen (or waiter and ladies)

The "missing dollar riddle" or "missing dollar paradox" is a famous puzzle that appears in almost every published collection of mathematical problems. The riddle begins with the story of three men who check into a hotel. The cost of their room, they are told, is $30. So, they each contribute $10 and go upstairs. Later the manager realizes that he has overcharged the men and that the actual cost should have been only $25. The manager promptly sends the bellboy upstairs to return the extra $5 to the men. The bellboy, however, decides to cheat the men and pockets $2 for himself and returns $1 to each of the men. As a result, each man has now paid $9 to stay in the room ($3 × $9 = $27) and the bellboy has pocketed $2 ($27 + $2 = $29). The men initially paid $30, so the question is where is the missing dollar?

Another version of this riddle changes the scene and the players—three ladies go to a restaurant for a meal. They receive a bill for $30. They each put $10 on the table, which the waiter collects and takes to the till. The cashier informs the waiter that the bill should only have been for $25 and returns $5 to the waiter in $1 coins. On the way back to the table the waiter realizes that he cannot divide the coins equally between the ladies. As they did not know the total of the revised bill, he decides to put $2 in his own pocket and give each of the ladies $1. Now that each lady has been given a dollar back, each of the ladies has paid $9. Three times 9 is 27. The waiter has $2 in his pocket. Two plus 27 is $29. The ladies originally handed over $30. Where is the missing dollar?

Although the setting and the characters has changed, what has not is the numbers—and the numbers are problematic in their compatibility. That is to say, the incorrect calculation brings us very close ($29) to the given initial value ($30), and that is where the problem, and the perceived paradox, lies. A variety of experts on a variety of websites and forum discussions have tried to explain the mis-calculation. We would like to clarify it as well. However, unlike other explanations, which stay with the story, we alter the story by implementing a numerical change. The paradox in the aforementioned situations is created by adding the $2 pocketed by the waiter or the bellboy to the $27 paid by the ladies or the men. Adding these two amounts does not answer any question. However, subtracting 2 from the 27 answers the question of how much was actually received as payment by the cashier or the receptionist at the hotel desk.

It is clear that the above explanation, or others similar to it, do not "work". People are still puzzled with the difference between the $29 that the story mentions and the desired initial $30, and so the search for the missing dollar continues. This is why the puzzle has survived for so many generations and, we suspect, will continue to intrigue curious minds for many generations to come. For those who strive to understand, however, we offer a different story—that is actually the same story but with different numbers. Let's say the room cost only $20, and the bellboy was sent to return $10 to the men. For simplicity of division, he pocketed $1 and returned $3 to each of the men. In this situation the men paid $7 each, for the total of $21. The bellboy has $1. Adding the actual payment to the one pocketed dollar gives us $22. Would it make sense to suggest, starting with the initial collection of $30, that $8 is missing? And if this is not convincing enough, let us change the numbers in the story once again, giving the men a "Stay with us for 1/3 the price" coupon, and send the bellboy to return to them $20. By now, knowing the bellboy's desire for a simple and fair division, we have him pocket $2 and return $18 to the men, $6 each. In this situation the men paid $4 each, for a total of $12. The bellboy has $2. Adding the actual payment to the two pocketed dollar gives us $14. Would it make sense to suggest, starting with the initial collection of $30, that $16 is missing?

We noticed that varying numbers, whether large or small, helps in making sense of the situation. Numerical variation in the story could be more convincing than any attempts to explain the original one. The absurdity of the missing dollar in the original situation is brought to surface when we establish the general structure of adding the paid amount to the pocketed amount. If the general structure of "missing money" makes no sense, neither does its specific example of the "missing dollar".

Numerical variation is recognized in instruction as a viable strategy; however, it is normally implemented starting with small or compatible numbers and then, once the structure is established, moved towards larger or stranger numbers. In this section we have also considered the benefits of numerical variation in the "opposite direction" in order to reveal the underlying general structure.

6 Pivotal examples as counterexamples

In previous sections we suggested how attentive choices of examples can help students to make appropriate generalizations. In this section we focus on the role of examples in refuting students' inappropriate generalizations. Zazkis and Chernoff (2006) introduced the notion of a *pivotal example*, defining it as an example that creates or resolves a cognitive conflict and makes learners change their mind with respect to a previously held strategy or belief. They further noted that a counterexample, while sufficient to refute a statement from a mathematical perspective, does not necessarily have the power needed to convince someone to abandon a previously made generalization. As such, a *pivotal example* is needed.

In order to introduce the concept of pivotal example, we describe the conversation with Tina who was articulate in her robust ideas and her struggle with disconfirming evidence. Tina was a preservice elementary school teacher in her early thirties enrolled in a course "Designs for Learning: Elementary Mathematics". This is a "methods" course in which students examine topics from the elementary school curriculum with the double purpose of (1) exposure to different pedagogical approaches and (2) opportunity to strengthen and enrich their own mathematics. In Shulman's terms (1986), acquiring curricular knowledge in this course serves as a vehicle to both develop pedagogical content knowledge and strengthen subject matter knowledge. Tina had extensive experience working in a tutoring centre prior to entering the teacher education program. She often shared with her classmates her experience with young learners using, what she referred to as, "tricks of the trade." Overall, she had a positive attitude towards mathematics and teaching mathematics and she was respected by her classmates as a source of ideas. However, at times she appeared limited by her own experiences and was not sufficiently open to new ideas and strategies presented in the course.

The topic of fractions was addressed in the course in considerable detail, as the related concepts are known to be problematic to both young learners and preservice elementary school teachers. One classroom session focused on a variety of ways to compare fractions, such as "bench mark," "compliment to a whole," "common numerator" strategy, etc. The goal in the presented comparison tasks was to avoid the "common denominator" strategy that students were already familiar with whenever possible. Towards the end of this session Tina approached the instructor and introduced a "different strategy."

> Tina: There is another strategy that you didn't mention, that has always worked for me.
> Instructor: OK, please show me.
> Tina: You simply take away the top from the bottom and see what is larger. Where the number is larger, the fraction is smaller, like 2/7 and 3/7, 5 is greater than 4, so this fraction (pointing to 2/7) is smaller.
> Instructor: Hmm, interesting …
> Tina: And the examples you showed work like that.
> Instructor: Would you explain *why* this works?
> Tina: I'm not sure how to explain this, it just makes sense.

Rather than acknowledging the strategy that Tina introduced as wrong, the instructor sought examples that demonstrate the discrepancy. However several examples discussed in that classroom session confirmed her strategy and seeking explanation as to why her strategy "works" appeared to not be of interest to her. Tina said it just "makes sense," and if something "works" as well as "makes sense," presumably, no explanation is needed. The excerpt with Tina demonstrates the "danger" of empirical generalization based on a limited example space. In what follows we describe the instructor's attempt to raise Tina's awareness of a possible discrepancy.

> Instructor: And how about different denominators?
> Tina: Oh-yeah, it will work, it always did.
> Instructor: So how about 1/2 and 2/4? Using your method we would conclude that one of these fractions is larger than another.
> Tina: But they never give you fractions that are the same to compare. So the method works when they are not the same.
> Instructor: And how about 5/6 and 6/7? We have just shown how to think of them and compare without finding a common denominator. How could you apply your method in this case?
> Tina: You can't if the difference is the same. But if it is not the same, it works [pause], I think it works, it always worked for me, in school, I mean. Like 4/9 and 5/7. You said, use 1/2 as a bench mark. I just looked at 5 here and 2 here [pointing at 4/9 and 5/7] and where you get 2 the fraction is larger.
> Instructor: And how about something like 9/10 and 91/100?
> Tina: [pause]. So are you saying that with ridiculously large number of pieces this doesn't work?
> Instructor: I'm just asking questions…

We note that, having faced a counterexample, Tina's immediate tendency was to amend her strategy, rather than to abandon it. In the above excerpt, presented with disconfirming evidence of 1/2 and 2/4, Tina reduced the scope of applicability of her method, claiming "the method works when they [i.e., factions] are not the same" and that her strategy cannot be used "if the difference is the same". To support her strategy, she immediately introduced another confirming example of 4/9 and 5/7. However, Tina's reaction to the example of 9/10 and 91/100 can be seen as recognition of the fact that her method was not applicable for the suggested case. It also can be seen as yet another attempt of reducing the scope of applicability. What is implicit in her words is the belief that the strategy "works" with "reasonable" numbers and not with "ridiculously large" ones. The disappointment in Tina's voice at this point was evident, but hard to convey in writing. It may, in fact, have not been the last example which made Tina reconsider her strategy, but rather the last example in conjunction with all the previous examples that she tried to dismiss. At this point the instructor turned to the class with the request to examine the strategy presented by Tina. This resulted in the presentation of numerous counterexamples, some of which were generated by Tina herself. Some examples—that included comparing 2/3 with 5/7, or 3/4 with 8/11—appeared to be more convincing than the initially suggested 9/10 and 91/100. Likely, these examples fall into the category of "normal" numbers rather than "ridiculously large" ones. However, rather than relative size of numbers, how can pivotal examples be characterized? We suggest that pivotal examples should fit within the learner's example spaces (Watson & Mason, 2005). Without such a fit a counterexample may merely be considered as a special case or an outlier, rather than as evidence that disconfirms a generalization.

The notion of example space has been central in our discussion. We have shown that in order for a counterexample to act to dismiss a mis-generalization, rather than be seen as an exception, a pivotal example should fit within the individual's example space. In what follows we demonstrate how a limited example space may lead to incorrect generalization.

7 Potential pitfalls in choice of examples

While generalizations are made by considering examples, there is no definite answer as to what kind of examples, and how many examples, are necessary in order to form a generalization. In a similar way that examples may lead to appropriate generalizations, they may also lead to a mis-generalizations. In this section we present two cases in which the choice of examples led to mis-generalization.

7.1 Teacher's examples: array of toothpicks

In this section we revisit the toothpick problem reported in Simmt, Davis, Gordon and Towers (2003). The problem invited students to consider a rectangular array of squares made out of toothpicks and provide an algebraic generalization to determine the number of toothpicks that are needed in constructing an $n \times k$ array. To guide students towards an appropriate algebraic expression they were presented with an example of a 3×6 array and asked to determine the number of toothpicks in the picture (see Fig. 1). The students were then asked to write a rule (that does not require counting each and every toothpick) for determining how many toothpicks there would be in a rectangular shape of any size (length and width) made with toothpicks, to express the rule algebraically, and to test their rule by determining the number toothpicks in a 10×15 rectangular array.

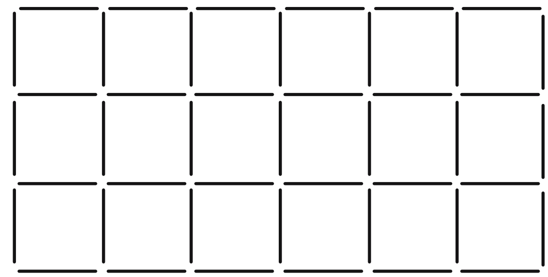


Fig. 1 Toothpicks in a rectangular array of squares

The number of toothpicks required to build a rectangular array of squares is given by $L(W + 1) + W(L + 1)$, where L and W stand for length and width respectively. However, considering the given 3×6 array, Arlene suggested that the number of toothpicks needed was 45, which is a result of $3^2 + 6^2$. Arlene generalized this observation, suggesting that the number of toothpicks in the general structure should be $L^2 + W^2$. She further applied her formula for the 10×15 array, achieving a correct result of 325.

Arlene's teacher was surprised by her solution, as an apparently incorrect generalization led to a correct result. The teacher's curiosity led to an examination of cases in which $L(W + 1) + W(L + 1)$ equals $L^2 + W^2$, as the equality does not hold in a general case. Simmt et al. (2003) presented the work of this teacher's specializing and generalizing which, in the end, led to the conclusion that the equality holds if L and W are consecutive triangular numbers[1].

The choice of numbers on the assignment was coincidental and in this particular case uncovered interesting mathematics. However, many coincidental choices may lead to inappropriate generalizations. Coincidental choices cannot be avoided. However, a teacher's awareness of how examples influence generalization is essential. Arlene's aforementioned generalization is an example of what Hewitt (1992) described as "train spotting" in that the generalization is based on fitting numbers rather than on some logical derivation considering where the numbers are coming from. Rather than just looking at a pattern of

[1] Triangular numbers (1, 3, 6, 10, 15, 21…) are numbers of the form $n(n+1)/2$, that are sums of consecutive natural numbers starting with 1; the name comes from considering the number of "dots" needed to draw a triangle.

numbers, and testing the suggested rule with a single example, it might be beneficial to guide students to connect their method of counting to a numerical rule as well as to check their suggested generalization choosing what Mason, Burton and Stacey (1985) refer to as "judicious examples".

7.2 Students' examples: counting squares

Edwards and Zazkis (2002) investigated students' work on the task of counting the number of squares crossed by a diagonal in a rectangular grid presented in Fig. 2.

The most general, single rule solution is given by $d = n + k\text{-GCD}(n,k)$ (where d is the number of squares crossed by the diagonal of an $n \times k$ rectangle, and GCD is the greatest common divisor).

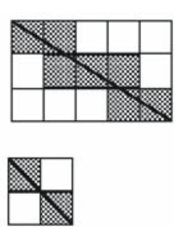
Examining the problem solving journals of preservice elementary school teachers, Edwards and Zazkis (2002) found that a popular solution was to claim that $d = n + k - 1$. In fact, even students who arrived at a complete and correct solution used this conjecture at some point in their investigation. For example, given a rectangle five squares wide and four squares tall, the diagonal would cross $5 + 4 - 1 = 8$ squares. As is clear from the general solution, this conjecture is true only when n and k are relatively prime. If the examples that students generated in search of a conjecture happened to have dimensions that are relatively prime, then this conjecture would appear to be supported.

Trying additional examples, the following solution was another popular generalization among the students: If n and k are both even, then $d = n + k - 2$; alternatively (that is, if both odd or one odd and one even) $d = n + k - 1$. Yet again, this emphasizes the choice of examples and the

Diagonals in a Rectangle

On squared paper draw a rectangle and draw in a diagonal.

How many grid squares are crossed by the diagonal?



In case of a 3x5 rectangle or a 2x2 rectangle above, we can simply count.

However, can we make a decision about a 100×167 or a 3600×288 rectangle?

In general, given n×k rectangle, how many grid squares are crossed by its diagonal?

Fig. 2 Diagonal in a rectangular array

dangers in attending to limited example spaces when making generalization.

Another issue of attention is the fact that even students who arrived at a correct and complete general solution were not satisfied. Their intention was to find a "formula" that depends only on n and k. The interference of gcd(n,k) in their solution appeared inconsistent with their concept image of a "general solution."

While we accept that some pitfalls in generalization are unavoidable, we believe that appropriate instructional attention could guide learners towards purposeful choices in numerical variation. Such choices will assist in generalizing based on recognition of structure rather than on "train spotting." This approach will help students progress from empirical to theoretical generalization (Dörfler, 1991) or from factual to contextual generalization (Radford, 2003).

8 Conclusion

"The reason for specializing is to permit and to promote generalizing" (Mason, 1999, p. 22). We agree with this general suggestion, but do so with caution. Specializing involves consideration of particular cases, in other words, consideration of examples. The caution is needed because not every set of examples will lead to a successful generalization. Particular features of examples are more helpful than others as a means towards recognizing and attending to the general structure. In this article we attempted to reveal a few of these features.

We discussed the connection between examples and generalization along several different avenues: the choice of examples that promote successful generalization and the choice of examples that help in refuting incorrect generalizations. We also attended to the potential pitfalls in considering a limited example space, either by "random" choices or by convenient choice of small numbers.

Our overall conclusion is that generalization is supported by enriching or explicating an individual's example space. Numerical variation serves as the means towards this end. Big numbers may bridge the gap between concrete small numbers and abstract algebraic symbolism. We demonstrated this in the conversation with Julie and her attempts to describe a pattern in standard algebraic notation. On the other hand, small or compatible numbers may help in revealing structure that is concealed when considering bigger numbers. We illustrated this in the cases of division by fraction and percentage increases that are over 100%. Further, simply a different choice of numbers may help in resolving strange coincidences and misleading arguments. We discussed this considering the hen and the egg riddle, the missing dollar paradox, the toothpick array, and the diagonal line problem.

Consistent with “the discipline of noticing”, “the purpose of the report is not to prove or persuade, but to suggest potential, to provide access for the reader to notice similar situations in their own practice and to demonstrate the range of work undertaken in order to reach the proposals being made” (Mason, 2002, p. 195). Having illustrated the importance of examples in forming and refuting generalization, we propose that it is the teacher’s role, not only to seek appropriate examples, but also to develop specializing strategies among learners. As such, our article contributes several examples for potential instructional implementation.

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