Does Understanding the Equal Sign Matter? Evidence from Solving Equations

Eric J. Knuth and Ana C. Stephens
University of Wisconsin—Madison

Nicole M. McNeil
Notre Dame University

Martha W. Alibali
University of Wisconsin—Madison

Given its important role in mathematics as well as its role as a gatekeeper to future educational and employment opportunities, algebra has become a focal point of both reform and research efforts in mathematics education. Understanding and using algebra is dependent on understanding a number of fundamental concepts, one of which is the concept of equality. This article focuses on middle school students' understanding of the equal sign and its relation to performance solving algebraic equations. The data indicate that many students lack a sophisticated understanding of the equal sign and that their understanding of the equal sign is associated with performance on equation-solving items. Moreover, the latter finding holds even when controlling for mathematics ability (as measured by standardized achievement test scores). Implications for instruction and curricular design are discussed.

Key words: Algebra, Learning, Middle grades 5–8, Prealgebra

Algebra continues to be a significant focus of both reform efforts (e.g., Lacampagne, Blair, & Kaput, 1995; National Council of Teachers of Mathematics [NCTM], 2000) and research (e.g., Bednarz, Kieran, & Lee, 1996; Kaput, Carraher, & Blanton, in press; Kieran, 1992; Olive, Izsak, & Blanton, 2002; RAND Mathematics Study Panel, 2003; Wagner & Kieran, 1989) in mathematics education. Such attention comes largely in response to growing concerns about students' inadequate understandings of and preparation in algebra as well as in recognition of the role that algebra plays as a gatekeeper to future educational and employment opportunities (Ladson-Billings, 1998; Moses & Cobb, 2001; National Research Council [NRC], 1998). Algebra reform, however, involves more than simply "fixing" 1st-year algebra courses. Indeed, there is an emerging consensus that algebra reform requires reconceptualizing the nature of algebra in school mathe-

matics, with many mathematics educators advocating that algebra be treated as a continuous K–12 strand (e.g., Carpenter & Levi, 1999; Kaput, 1998; NCTM, 2000). "By viewing algebra as a strand in the curriculum from prekindergarten on, teachers can help students build a solid foundation of understanding and experience as a preparation for more-sophisticated work in algebra in the middle grades and high school" (NCTM, 2000, p. 37).

One concept that is fundamental to algebra understanding and that has received considerable research attention is that of equality and, in particular, understanding of the equal sign (e.g., Alibali, 1999; Behr, Erlwanger, & Nichols, 1980; Falkner, Levi, & Carpenter, 1999; Kieran, 1981; McNeil & Alibali, 2005). The ubiquitous presence of the equal sign at all levels of mathematics highlights its importance. The concept of equality and its symbolic instantiation are traditionally introduced during students' early elementary school education, with little instructional time explicitly spent on the concept in the later grades. Yet, as the RAND Mathematics Study Panel (2003) contended, "the notion of 'equal' is complex and difficult for students to comprehend" (p. 53). Studies of students' understanding and use of equality (and the equal sign) lend support to this contention (e.g., Alibali, 1999; Behr et al., 1980; Falkner et al., 1999; Kieran, 1981; McNeil & Alibali, 2005). In this article, we report results from a study[1] that examined middle school students' understanding of the equal sign and its relation to their performance solving algebraic equations.

STUDENT UNDERSTANDING OF THE EQUAL SIGN

Many elementary and middle school students demonstrate inadequate understanding of the meaning of the equal sign, frequently viewing the symbol as an announcement of the result of an arithmetic operation rather than as a symbol of mathematical equivalence (Baroody & Ginsburg, 1983; Behr et al., 1980; Kieran, 1981; Rittle-Johnson & Alibali, 1999). Such an "operational" view is consistent with equal sign definitions—including "the total" or "the answer"—generated by third-through fifth-grade students in McNeil and Alibali's (2005) study. This view is also consistent with first- through sixth-grade students' attempts to solve the equation $8 + 4 = \square + 5$ (Falkner et al., 1999) and third- and fourth-grade students' attempts to solve equations such as $4 + 3 + 5 = \_\_ + 5$ (e.g., Alibali, 1999). Falkner et al. found that many students provided answers of 12, 17, or 12 and 17—answers that are consistent with an understanding of the equal sign as announcing a result. Similarly, Alibali found that many students added all the numbers in the equation or added all the numbers before the equal sign, again indicating an operational view of the equal sign.

It has been suggested that this well documented (mis)conception might be due, at least in part, to students' elementary school experiences (Baroody & Ginsburg, 1983; Behr et al., 1980; Carpenter, Franke, & Levi, 2003; Seo & Ginsburg, 2003).

---

[1] The study is part of a 5-year longitudinal study seeking to understand the development of middle school students' algebraic reasoning.

Although viewing the equal sign as a "do something signal" (Kieran, 1981, p. 319) is generally not problematic when solving "typical" elementary school arithmetic problems of the form $a + b = \square$, such a view may not serve students well when they encounter more complex equations in later grades. In fact, many of the difficulties that students have when working with symbolic expressions and equations may be attributed to their misconceptions about the meaning of the equal sign. Kieran, for example, proposed that long-standing misconceptions about the meaning of the equal sign might be the root cause of high school students' difficulties dealing with polynomial expressions. She found that 12- and 13-year-olds had difficulty assigning meaning to expressions such as $3a$, $a + 3$, and $3a + 5a$ because, as one student stated, "There is no equal sign with a number after it" (p. 324). Carpenter et al. (2003) questioned the meaning such students might make of the procedures one might use to solve equations such as $5x + 32 = 97$ (starting with subtracting 32 from each side resulting in $5x + 32 - 32 = 97 - 32$): "What kind of meaning can students who exhibit … misconceptions of the equal sign … attribute to this equation?" (p. 22). Students must understand the equal sign as expressing a relation in order to make sense of the transformations performed on such an equation.

In sum, the literature highlights the misconceptions that many students possess regarding the meaning of the equal sign. Such misconceptions persist among some high school and even college students (e.g., McNeil & Alibali, 2005; Mevarech & Yitschak, 1983), suggesting that they are robust and long lasting. Given the results of the aforementioned research, it seems reasonable to hypothesize that there is a relation between students' understanding of the equal sign and their success working with symbolic expressions and algebraic equations. Yet evidence that such a relation does in fact exist is lacking. The goal of this article is to present evidence regarding the relation between students' understanding of the equal sign and their success solving traditional algebraic equations—in essence, to show that understanding the equal sign does matter.

METHOD

Participants

Participants were 177 middle school (47 sixth grade, 72 seventh grade, 58 eighth grade) students drawn from an ethnically diverse middle school in the American Midwest. The demographic breakdown of the school's student population is as follows: 25% African American, 5% Hispanic, 7% Asian, and 62% White. The middle school had recently adopted the reform-based curricular program Connected Mathematics; at the time of this study, the sixth-grade teachers were in their 2nd year of implementation, while the seventh- and eighth-grade teachers were in their 3rd year of implementation.[2] In addition, with the exception of one section of eighth-grade algebra, the classes were not tracked (e.g., all seventh-grade students were in the same mathematics course).

---

[2] Before adopting the Connected Mathematics curriculum, the middle school teachers used a variety of textbook series, often differing from grade level to grade level and even differing within a grade level.

Given the focus of this article, it should be noted that the authors of the Connected Mathematics curriculum suggest that the development of algebra begins in the sixth-grade units, with increasing attention being given to algebra in the seventh- and eighth-grade units (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2002). Of particular relevance to this study, traditional algebraic topics such as solving linear equations receive explicit attention beginning in seventh grade.

Data Collection

The data that are the focus of this article consist of students' responses to items from a written assessment that targeted their understandings of various aspects of algebra. The assessment was designed by the authors, and it included a mixture of items drawn from previous studies as well as items developed specifically for the assessment. There were two alternate forms of the assessment, which varied only in specific numbers or the problem format used in some of the items (i.e., verbal format versus symbolic format). The two forms were randomly assigned to students in each grade, and all students completed the entire assessment. This article addresses students' responses to three items.

One item required students to interpret the equal sign (see Figure 1), and two items required students to determine the solution to an algebraic equation (see Figure 2). Each student received the equal sign item and one of the two equation items. The equal sign interpretation item required students to first name the equal sign symbol (first prompt), then provide a statement regarding the symbol's meaning (second prompt), and then, if possible, provide a statement regarding an alternative meaning (third prompt). The rationale for the first prompt was to preempt students from using the name of the symbol in their response to the second prompt (e.g., "the symbol means equal"). The rationale for the third prompt was based on our previous work, in which we found that students often offer more than one interpretation when given the opportunity. The equation-solving items required students to determine the solution to typical 1st-year algebra equations. We used the prompt "What value of m will make the following number sentence true?" for two reasons. First, we wanted to avoid running into problems asso-

---

The following questions are about this statement:

    3 + 4 = 7
         ↑

(a) The arrow above points to a symbol. What is the name of the symbol?

(b) What does the symbol mean?

(c) Can the symbol mean anything else? If yes, please explain.

Figure 1. Interpreting the equal sign.

What value of *m* will make the following number sentence true?

(a) $4m + 10 = 70$

(b) $3m + 7 = 25$

Figure 2. Equation solving items.

ciated with students misunderstanding the more prototypical (in terms of 1st-year algebra) prompt "Solve the equation." Second, we felt that the prompt we used would likely be more consistent with the prompt often used in elementary school problems (i.e., "What value will make the number sentence, $8 + \square = 12$, true?").

Data Analysis

In this section, we provide details regarding the coding of each item. For all three items, responses that students left blank or for which they wrote "I don't know" were grouped into a *no response/don't know* category, and response types that were not sufficiently frequent to warrant their own codes were grouped into an *other* category.

Coding equal sign definitions. Student responses to parts (b) and (c) of Item 1 were coded as *relational*, *operational*, *other*, or *no response/don't know*, with the majority of responses falling into the first two categories. A response was coded as *relational* if a student expressed the general idea that the equal sign means "the same as" and as *operational* if the student expressed the general idea that the equal sign means "add the numbers" or "the answer." The *other* category included definitions such as "it means equals" or "it means equal to" as well as direct translations of the problem statement, such as "3 plus 4 equals 7." In addition to coding responses to parts (b) and (c) separately, students were also assigned an overall code indicating their "best" interpretation. Many students provided two interpretations, sometimes one *relational* and one *operational*; in such cases, the responses were assigned an overall code of *relational*.

Coding equation-solving performance. For the equation-solving items, we coded both correctness and strategy. A response was coded as *correct* if the student identified the value of *m* that satisfied the equation, *incorrect* if the student provided a value for *m* that did not satisfy the equation, or *no response/don't know* if the student did not provide a value for *m*. Students' strategies were classified into one of the following six categories: *answer only*, *no response/don't know*, *guess and test*, *unwind*, *algebra*, and *other*. A response was coded as *answer only* if only a value of *m* was provided (i.e., no corresponding work was shown). A response was coded as *guess and test* if the student tested (substituted) various values of *m* in the equation until he or she arrived at the correct value. A response was coded as *unwind* if the student solved the problem by working backward through the constraints provided in the problem, in essence, "unwinding" the constraints by inverting

operations and performing arithmetic operations rather than using algebraic manipulation. It is important to note that in using an unwind strategy students start with the constant value from one side of the equation and then perform arithmetic operations on that value. For example, for equation (a) of Figure 2, a student using the unwind strategy would subtract 10 from 70 and then divide this result by 4 in order to determine the value of *m*. Finally, responses were coded as *algebra* if the student solved the equation using a typical algebraic method (i.e., performing the same transformations on each side of the equation).

Coding reliability. To assess reliability of the coding procedure, a second coder rescored approximately 20% of the data. Agreement between coders was 93% for coding students' interpretations of the equal sign, 100% for coding the correctness of students' responses to the equation-solving items, and 90% for coding students' strategies on the equation-solving items.

RESULTS

We focus first on students' interpretations of the equal sign, and then on how these interpretations relate to students' performance solving equations. In addition, for a subset of students we examine their interpretations of the equal sign and their performance solving equations in relation to their performance on the mathematics, reading, and language components of a national standardized test (Terra Nova for sixth- and seventh-grade students and Wisconsin Knowledge and Concepts Exam [WKCE] for eighth-grade students).[3] Where applicable, representative excerpts from students' written responses are provided to illustrate findings. In reporting the results, we describe and illustrate only those coding categories that are most germane to the focus of the article (e.g., the *other* category is not discussed in detail). Finally, the statistical analysis was performed using logistic regression because the outcome variables of interest were categorical. Full details of each logistic regression model are presented in the Appendix.

Equal Sign Interpretations

Table 1 presents the distribution of equal sign definitions as a function of grade. As seen in the table, the majority of sixth- and eighth-grade students provided definitions that were coded as operational, whereas substantially fewer sixth- and eighth-grade students provided definitions coded as relational. It is interesting that in seventh grade, slightly more students provided definitions coded as relational than provided definitions coded as operational; nevertheless, fewer than half of the seventh-grade students provided a relational definition. The following student responses are representative of responses coded as operational:

"What the sum of the two numbers are" (sixth-grade student).

---

[3] Of relevance to this study, the mathematics components of the standardized tests included items addressing traditional algebraic topics (e.g., solving two-step linear equations) at all three grade levels.

Table 1
Percent of Students at Each Grade Level Who Provided Each Type of Equal Sign Definition as Their Best Definition

Best Definition	Grade 6	Grade 7	Grade 8
Operational	53	36	52
Relational	32	43	31
Other	15	20	17
No response/don't know	0	1	0

"A sign connecting the answer to the problem" (seventh-grade student).

"What the problem's answer is" (seventh-grade student).

"The total" (eighth-grade student).

"How much the numbers added together equal" (eighth-grade student).

In contrast, the following student responses are representative of those coded as relational:

"It means that what is to the left and right of the sign mean the same thing" (sixth-grade student).

"The same as, same value" (seventh-grade student).

"The left side of the equals sign and the right side of the equals sign are the same value" (eighth-grade student).

"The expression on the left side is equal to the expression on the right side" (eighth-grade student).

We used logistic regression to examine the relation between grade level (6, 7, or 8) and the likelihood of exhibiting a relational understanding of the equal sign. It is surprising that students were not more likely to exhibit a relational understanding of the equal sign as grade level increased (i.e., there was no linear trend across grades), nor was there a U-shaped pattern in students' likelihood of exhibiting a relational understanding across the grade levels (i.e., there was no quadratic trend across grades; see the Appendix for details of the statistical analysis). Thus, there was no evidence in this data set to suggest that students' likelihood of exhibiting a relational understanding of the equal sign changes across the middle school grades. We also examined whether there were differences in students' equal sign understanding across teachers within each grade level; there were no significant teacher effects.

We next examined the relation between students' equal sign definitions and mathematics ability, as assessed using standardized tests. We used logistic regression to predict the log of the odds of exhibiting a relational understanding of the equal sign for the subset of students for whom we had standardized test scores ($N = 65$ students across the three grades). Predictor variables included grade level (6, 7, or 8) and students' national percentiles on the mathematics, reading, and language components of a national standardized test. Students with higher national percentile scores on the mathematics component of the standardized test were more likely to exhibit a relational understanding of the equal sign, $\hat{\beta} = 0.061$, $z = 2.44$, Wald $(1, N = 65) = 5.93$,

$p = .02$. However, students' likelihood of exhibiting a relational understanding of the equal sign was not related to national percentile score on the reading or language components of the standardized tests. Additionally, as in the overall sample, the effect of grade level was not significant in the subsample (neither the linear nor the quadratic trend). Thus, students' equal sign understanding was associated with mathematics ability, but not with reading ability, language ability, or grade level.

Relation to Equation-Solving Performance

We next examined the relation between students' equal sign understanding and their performance in solving algebraic equations. To address this issue, we examined two outcome measures: (1) whether or not students solved the equations correctly, and (2) whether or not students used an algebraic strategy to solve the equations. Overall performance and use of an algebraic strategy did not differ across the two equation-solving items; thus, we collapsed across items in the analyses presented here. Based on prior work (e.g., Kieran, 1981), we predicted that students who lack a relational understanding of the equal sign might have difficulty understanding the steps involved in an algebraic strategy (e.g., why do the same thing to both sides?). Consequently, we expected such students to use nonalgebraic strategies to solve the equations.

Correctness. Figures 3 and 4 present the proportion of students at each grade level and with each level of equal sign understanding who solved the equations correctly. We used logistic regression to predict the log of the odds of solving the equations correctly. Predictor variables included grade level (6, 7, or 8) and equal sign understanding (relational or not). Students were more likely to solve the equations correctly as grade level increased, $\beta = 1.06$, $z = 3.29$, Wald $(1, N = 177) = 10.81$, $p = .001$. More important, however, students who exhibited a relational understanding of the equal sign were more likely than students who did not exhibit a relational understanding to solve the equations correctly, $\hat{\beta} = -1.74$, $z = -4.76$, Wald
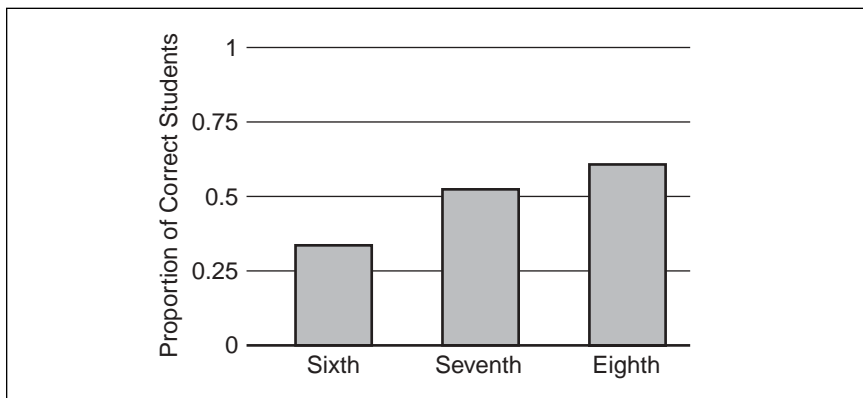

Figure 3. Proportion of sixth-, seventh-, and eighth-grade students who solved the equations correctly.

$(1, N = 177) = 22.64$, $p < .001$. As seen in Figure 4, at each grade level, a greater proportion of students who exhibited a relational understanding of the equal sign solved the equations correctly.

It might be argued that the relation between equal sign understanding and equation-solving performance is because of students' general abilities in mathematics and not because of a relation between equal sign understanding and equation solving per se. To address this issue, we performed a similar analysis on a subset of the students for whom we had standardized test scores ($N = 65$), so we could control for mathematics ability. In addition to grade level (6, 7, or 8) and equal sign understanding (relational or not), we included national percentile scores on the mathematics, reading, and language components of a national standardized test as predictors in the model. Three effects were significant: the linear effect of grade level, $\hat{\beta} = 7.17$, $z = 2.54$, Wald $(1, N = 65) = 6.43$, $p = .01$; the effect of equal sign understanding, $\hat{\beta} = -4.58$, $z = -1.96$, Wald $(1, N = 65) = 3.85$, $p = .05$; and the effect of national percentile on the mathematics component of the standardized test, $\hat{\beta} = 0.15$, $z = 2.10$, Wald $(1, N = 65) = 4.38$, $p = .04$. There were no effects of national percentile on the reading and language components of the standardized test.

Thus, even when controlling for grade level and standardized mathematics test scores, the association between equal sign understanding and equation-solving performance was significant. In other words, the observed relation between equal sign understanding and equation-solving performance was not simply because better students were performing well on both items.

Use of an algebraic strategy. Table 2 presents the distribution of strategies used by students in each grade to solve the equations. It is clear from the table that the proportion of student strategies coded as *answer only* or *no response/don't know* decreased with grade level (77% in sixth grade to 31% in eighth grade), perhaps
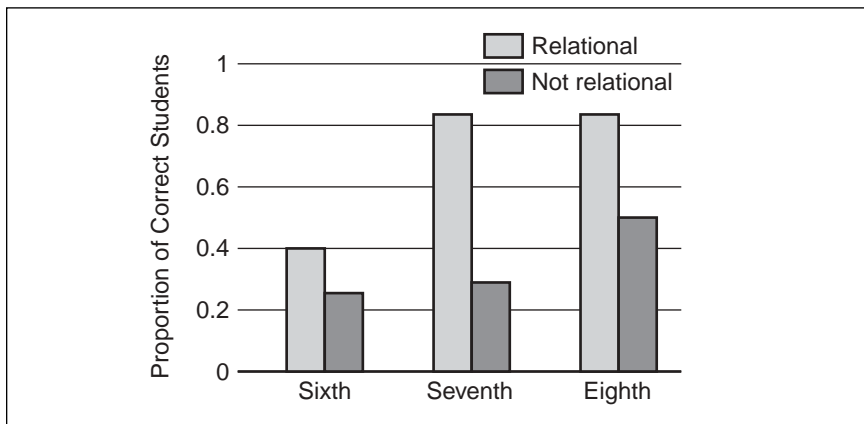

Figure 4. Proportion of sixth-, seventh-, and eighth-grade students in each equal sign understanding category who solved the equations correctly.

because students were gaining more experience with literal symbols and algebraic equations. The table also displays an increase in the proportion of seventh- and eighth-grade students using "prealgebraic" strategies (strategies coded as *guess and test* and *unwind*) as well as algebraic strategies (strategies coded as *algebra*). Similar to Kieran (1989), we consider *guess and test* to be an arithmetic strategy; we also consider *unwind* to be a prealgebraic strategy because this strategy does not emphasize the symmetry of an equation (in contrast to *algebra* strategies with their emphasis on performing the same operation on *both* sides of the equal sign). Representative examples of the prealgebraic strategies are displayed in Figure 5, and a representative example of an algebraic strategy is displayed in Figure 6. Table 3 displays the proportion of problem-solving strategies used by students at each grade level who solved the equations correctly. As seen in the table, the most effective strategy at Grade 6 was *guess and test*; at Grade 7, it was *unwind*; and at Grade 8, *algebra*.

None of the sixth-grade students and only one seventh-grade student used an algebraic strategy to solve the equations (see Table 2); thus, the relation between equal

Table 2
Percent of Students at Each Grade Level Who Used Each Type of Problem-solving Strategy on the Equation-solving Items

Strategy Code	Grade 6	Grade 7	Grade 8
Answer only	51	25	17
No response/don't know	26	24	14
Guess and test	6	15	16
Unwind	9	26	9
Algebra	0	1	33
Other	9	8	12

Figure 5. Examples of (a) *guess and test* and (b) *unwind* strategies.

sign understanding and use of an algebraic strategy could not be tested among sixth- and seventh-grade students. For the eighth-grade students, however, students who exhibited a relational understanding of the equal sign were more likely than those who did not exhibit a relational understanding to use an algebraic strategy (see Figure 7),

7. What value of *m* will make the following number sentence true?
$$4m + 10 = 70$$
$$-10 \quad -10$$
$$\frac{4m = 60}{4 \quad 4}$$
$$\boxed{m = 15}$$

Figure 6. Example of an *algebra* strategy

Table 3
Percent of Students Who Used Each Type of Problem-solving Strategy Who Arrived at the Correct Solution

Strategy Code	Grade 6	Grade 7	Grade 8	Total
Answer only	33% (24)	56% (18)	60% (10)	46% (52)
No response/don't know	0% (12)	0% (17)	0% (8)	0% (37)
Guess and test	100% (3)	73% (11)	78% (9)	78% (23)
Unwind	75% (4)	95% (19)	60% (5)	86% (28)
Algebra	—	0% (1)	100% (19)	95% (20)
Other	0% (4)	33% (6)	0% (7)	12% (17)

Note. The number in parentheses is the number of students who used the particular problem-solving strategy. In Grade 6, no students attempted an algebraic strategy.
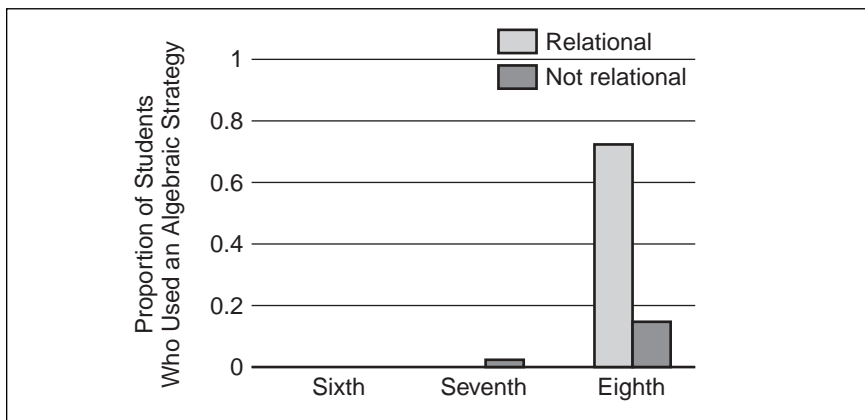

Figure 7. Proportion of sixth-, seventh-, and eighth-grade students in each equal sign understanding category who used an algebraic strategy to solve the equations.

χ^2 (1, $N = 58$) $= 18.45$, $p < .001$. It is important to note that all 19 eighth-grade students who used an algebraic strategy also solved the equations correctly.

We could not analyze the effects of mathematics ability on use of an algebraic strategy because there were too few eighth-grade students for whom we had standardized test scores. Recall, however, that some eighth-grade students were enrolled in an algebra course, and others were not. It seemed probable that students enrolled in algebra would be more likely both to use an algebraic strategy and to provide a relational definition of the equal sign. If this were the case, the observed relation between equal sign understanding and use of an algebraic strategy might be because both were being related to algebra course work rather than to a relation between equal sign understanding and use of an algebraic strategy per se. As expected, eighth-grade students enrolled in algebra ($N = 7$) were more likely than those not enrolled in algebra ($N = 45$) to use an algebraic strategy (86% versus 29% of students). In addition, students enrolled in algebra were more likely than students not enrolled in algebra to give a relational definition of the equal sign (71% versus 29% of students). However, the relation between equal sign understanding and use of an algebraic strategy remained significant even when students enrolled in algebra were excluded from the analysis, χ^2 (1, $N = 45$) $= 9.49$, $p = .002$. Thus, the observed relation between equal sign understanding and use of an algebraic strategy was not because students enrolled in algebra were performing well on both items. These data suggest that equal sign understanding informs students' use of an algebraic strategy.

UNDERSTANDING THE EQUAL SIGN DOES MATTER

In this study, we examined middle school students' understanding of the equal sign and how it relates to equation-solving performance. The results suggest that relatively few middle school students hold a relational view of the equal sign. Further, there was no evidence to indicate that equal sign understanding improves across the middle grades (although modest improvement across the middle grades has been reported elsewhere—see, for example, Knuth, Alibali, McNeil, Weinberg, & Stephens, 2005). As seen in Table 1, in the present study, although there were slight improvements from Grade 6 to Grade 7, performance declined in Grade 8. This differs from the pattern observed by Knuth et al. (2005), who noted improvements from Grade 7 to Grade 8 (as well as from Grade 6 to Grade 7). It is worth noting, however, that overall performance in this previous study remained relatively low (less than 50% of the eighth-grade students demonstrated a relational view of the equal sign). The reasons for eighth-grade students' particularly poor performance in the present study are unclear.

The generally poor performance on measures of equal sign understanding is perhaps not surprising, given the lack of explicit focus on the equal sign in middle school curricula. Nevertheless, these findings are cause for concern because we also found a strong relation between equal sign understanding and success in solving equations.

Kieran (1992) suggested that "one of the requirements for generating and adequately interpreting structural representations such as equations is a conception of the symmetric

and transitive character of equality—sometimes referred to as the 'left-right equivalence' of the equal sign" (p. 398). In support of this claim, we have documented elsewhere (Knuth et al., 2005) that such an understanding of the equal sign does indeed support a structural conception of equations. Specifically, we found a positive relation between equal sign understanding and performance on equivalent equations problems (e.g., judging that the value of m that satisfies the equation, $2m + 15 = 31$, will be the same value that satisfies the equation $2m + 15 - 9 = 31 - 9$). An important omission in Kieran's statement (as well as in the research literature), however, is that a relational view of the equal sign is necessary not only to meaningfully generate and interpret equations but also to meaningfully *operate* on equations. The results of this study support the latter point and suggest that efforts to enhance students' understanding of the equal sign may pay off in better performance in algebra.

We found a strong positive relation between middle school students' equal sign understanding and their equation-solving performance, and we showed that this relation holds even when controlling for mathematics ability (as assessed via standardized tests). This finding is noteworthy in that it suggests that even students having no experience with formal algebra (sixth- and seventh-grade students in particular) have a better understanding of how to solve equations when they have a relational understanding of the equal sign. In addition, we found a strong positive relation between equal sign understanding and use of an algebraic strategy among eighth-grade students (students who have had more experience with algebraic ideas and symbols as compared to their peers in sixth and seventh grade), and we showed that this relation holds for those eighth-grade students who were not enrolled in an algebra course. Taken together, these findings suggest that understanding the equal sign is a pivotal aspect of success in solving algebraic equations (whether using an algebraic strategy or not). These findings also help build a case for the importance of continuing to explicitly develop students' understanding of the equal sign during their middle school mathematics education.

Why might middle school students hold an operational view of the equal sign? The equal sign is traditionally introduced during students' early elementary years, with little instructional time explicitly spent on the concept in the later grades. In fact, teachers generally assume that once students have been introduced to the concept during elementary school, little or no review is needed. The lack of explicit attention to the equal sign in the later grades may explain, in large part, why many students continue to show inadequate understandings of its meaning in middle school and beyond (e.g., McNeil & Alibali, 2005; Mevarech & Yitschak, 1983). Further exacerbating students' lack of opportunities to develop their understanding of the equal sign is the fact that very little attention is paid to the symbol in curricular materials—despite its ubiquitous presence. Moreover, analyses of middle school curricular materials suggest that relational uses of the equal sign are less common than operational uses (McNeil et al., in press). This pattern of exposure may actually condition students to favor less sophisticated and generalized uses of equivalence (such as "operations equals answer").

In recent years, mathematics educators have recognized the importance of continuing to explicitly develop students' understanding of equality and of the equal sign,

in particular (see, for example, Carpenter et al., 2003). These efforts, however, have primarily concentrated on the elementary school grades. We argue that there is a clear need for continued attention to be given to the notion of equality in the middle school grades. In our professional development work with middle school teachers, for example, we encourage teachers to look for opportunities to engage students in conversations about the equal sign during classroom interactions as well as to create such opportunities intentionally. Anyone who has spent time in mathematics classrooms has probably witnessed the equality "strings" students often produce (e.g., $3 + 5 = 8 + 2 = 10 + 5 = 15$); these equality strings provide an excellent opportunity to discuss with students the meaning of the equal sign and its proper use. Even providing students with equations to solve (arithmetic or algebraic) in which numbers and operations (or symbolic expressions and operations) appear on both sides of the equal sign may help promote more appropriate interpretations and uses of the equal sign.

CLOSING REMARKS

Algebra continues to be a struggle for many students—a fact that has led to 1st-year algebra courses in the United States being characterized as "an unmitigated disaster for most students" (NRC, 1998, p. 1). Although students' difficulties learning algebra have been attributed to a variety of factors, we agree with Carpenter et al.'s (2003) contention that a "limited conception of what the equal sign means is one of the major stumbling blocks in learning algebra. Virtually all manipulations on equations require understanding that the equal sign represents a relation" (p. 22). The results of our study lend support to this contention: Students who understand the equal sign as a relational symbol of equivalence are more successful solving algebraic equations than their peers who do not have such an understanding. This finding, coupled with the fact that, overall, far too few middle school students viewed the equal sign as representing a relation, clearly illustrate the need to give more explicit attention to the equal sign in middle school mathematics. As NCTM (2000) recommended, "The notion of equality [and its symbolic representation] also should be developed throughout the curriculum" (p. 39). This recommendation, if followed, may lead to success in learning algebra by greater numbers of students.

REFERENCES

Alibali, M. W. (1999). How children change their minds: Strategy change can be gradual or abrupt. *Developmental Psychology, 35*, 127–145.

Baroody, A., & Ginsburg, H. (1983). The effects of instruction on children's understanding of the "equals" sign. *Elementary School Journal, 84*, 199–212.

Bednarz, N., Kieran, C., & Lee, L. (1996). *Approaches to algebra: Perspectives for research and teaching*. Dordrecht, The Netherlands: Kluwer Academic Press.

Behr, M. J., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching, 92*, 13–15.

Carpenter, T. P., Franke, M. L., & Levi, L. (2003). *Thinking mathematically: Integrating arithmetic and algebra in the elementary school*. Portsmouth, NH: Heinemann.

Carpenter, T. P., & Levi, L. (1999, April). *Developing conceptions of algebraic reasoning in the*

primary grades. Paper presented at the annual meeting of the American Educational Research Association, Montreal, Canada.

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics, 6*, 56–60.

Kaput, J. (1998, May). *Transforming algebra from an engine of inequity to an engine of mathematical power by "algebrafying" the K–12 curriculum*. Paper presented at the Algebra Symposium, Washington, DC.

Kaput, J., Carraher, D., & Blanton, M. (In press). *Algebra in the early grades*. Mahwah, NJ: Erlbaum.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics, 12*, 317–326.

Kieran, C. (1989). The early learning of algebra: A structural perspective. In S. Wagner & C. Kieran (Eds.), *Research issues in the learning and teaching of algebra* (pp. 33–56). Reston, VA: NCTM.

Kieran, C. (1992). The learning and teaching of school algebra. In D. Grouws (Ed.), *Handbook of research on mathematics teaching and learning* (pp. 390–419). New York: MacMillan.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. S. (2005). Middle school students' understanding of core algebraic concepts: Equality & variable. *Zentralblatt für Didaktik der Mathematik* [International Reviews on Mathematical Education], *37*, 68–76.

Lacampagne, C., Blair, W., & Kaput, J. (1995). *The algebra initiative colloquium*. Washington, DC: US Department of Education & Office of Educational Research and Improvement.

Ladson-Billings, G. (1998). It doesn't add up: African American students' mathematics achievement. In C. Malloy and L. Brader-Araje (Eds.), *Challenges in the mathematics education of African American children: Proceedings of the Benjamin Banneker Association Leadership Conference* (pp. 7–14). Reston, VA: National Council of Teachers of Mathematics.

Lappan, G., Fey, J., Fitzgerald, W., Friel, S., & Phillips, E. (2002). *Getting to know Connected Mathematics: An implementation guide*. Glenview, IL: Prentice Hall.

McNeil, N. M., & Alibali, M. W. (2005). Knowledge change as a function of mathematics experience: All contexts are not created equal. *Journal of Cognition and Development, 6*, 285–306.

McNeil, N. M., Grandau, L., Knuth, E. J., Alibali, M. W., Stephens, A. S., Hattikudur, S., et al. (In press). Middle-school students' understanding of the equal sign: The books they read can't help. *Cognition & Instruction*.

Mevarech, Z., & Yitzchak, D. (1983). Students' misconceptions of the equivalence relationship. In R. Hershkowitz (Ed.), *Proceedings of the Seventh International Conference for the Psychology of Mathematics Education* (pp. 313–318). Rehovot, Israel: Weizmann Institute of Science.

Moses, R., & Cobb, C. (2001). *Radical equations: Math literacy and civil rights*. Boston, MA: Beacon Press.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Research Council. (1998). *The nature and role of algebra in the K–14 curriculum*. Washington, DC: National Academy Press.

Olive, J., Izsak, A., & Blanton, M. (2002). Investigating and enhancing the development of algebraic reasoning in the early grades (K–8): The Early Algebra Working Group. In D. S. Mewborn, P. Sztajn, D. Y. White, H. G. Wiegel, R. L. Bryant, & K. Nooney (Eds.), *Proceedings of the Twenty-fourth annual meeting of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 119–120). Columbus, OH: ERIC.

RAND Mathematics Study Panel. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education*. Santa Monica, CA: RAND.

Rittle-Johnson, B., & Alibali, M. W. (1999). Conceptual and procedural understanding: Does one lead to the other? *Journal of Educational Psychology, 91*, 175–189.

Seo, K-H, & Ginsburg, H. P. (2003). "You've got to carefully read the math sentence . . .": Classroom context and children's interpretations of the equals sign. In A. J. Baroody & A. Dowker (Eds.), *The development of arithmetic concepts and skills*. Mahwah, NJ: Erlbaum.

Wagner, S., & Kieran, C. (1989). *Research issues in the learning and teaching of algebra*. Reston, VA: NCTM.

Authors

Eric J. Knuth, Associate Professor, Department of Curriculum and Instruction, University of Wisconsin, Madison, WI 53706; knuth@education.wisc.edu

Ana C. Stephens, Assistant Researcher, Wisconsin Center for Education Research, University of Wisconsin, Madison, WI 53706; acstephens@wisc.edu

Nicole M. McNeil, Assistant Professor, Department of Psychology, University of Notre Dame, Notre Dame, IN 46556; nmcneil@nd.edu

Martha W. Alibali, Professor, Department of Psychology, University of Wisconsin, Madison, WI 53706; mwalibali@wisc.edu

APPENDIX

Details of Each Logistic Regression Model

Table A1
Dependent Response: Whether or Not a Relational Understanding Is Exhibited

Effect	β	z	Wald	p
	Model 1 ($N = 177$)			
Grade level—linear	–0.03	–0.10	0.009	.92
Grade level—quadratic	–0.41	–1.57	2.45	.12
	Model 2 ($N = 65$)			
National percentile—math	0.061	2.44	5.93	.02
National percentile—reading	–0.001	–0.04	0.002	.97
National percentile—language	–0.01	–0.38	0.15	.70
Grade level—linear	–0.71	–1.25	1.58	.21
Grade level—quadratic	–0.08	–0.16	0.025	.87

Note. In each regression model, Y = 1 if relational and Y = 0 if not relational.

Table A2
Dependent Response: Whether or Not Equation Is Solved Correctly

Effect	β	z	Wald	p
	Model 1 ($N = 177$)			
Grade level—linear	1.06	3.29	10.81	.001
Grade level—quadratic	–0.14	–0.51	0.26	.61
Equal sign understanding	–1.74	–4.76	22.64	<.001
	Model 2 ($N = 65$)			
National percentile—math	0.15	2.10	4.38	.04
National percentile—reading	–0.11	–1.26	1.59	.21
National percentile—language	0.13	1.65	2.74	.10
Grade level—linear	7.17	2.54	6.43	.01
Grade level—quadratic	–0.82	–0.63	0.39	.53
Equal sign understanding	–4.58	–1.96	3.85	.05

Note. In each regression model, Y = 1 if correct and Y = 0 if incorrect.