

Technology and teaching of mathematics

Work on this topic comprised two parts, the first related to the use of the Geogebra software for the teaching of algebra and functions, the second related to probability and the use of Maple software. The majority of the participants had not used either of these programs before so the meetings combined mathematical work, didactic reflection and initiation guided by this software. The topic was led by Morou Amidou (Niger) and Moustapha Sokhna (Senegal) for the Geogebra meetings and by Morou Amidou and Pierre Arnoux for the probability section.

Transverse topics corresponding to regional priorities

Four topics were selected for this part: local numbering systems and their influence on the teaching of number and operations in the region, teaching with large groups of pupils, the evolution of curriculum reforms involving the competency approach, and taking multilingualism into account in the teaching of mathematics. The discussions on each topic were prepared and controlled by Kalifa Traore, Patricia Nebout, Mustapha Sokhna, Sidi Bekaye Sokona, Mamadou S. Sangaré and Mamadou Kanouté.

In all sections, and in keeping with the philosophy of “practising what we preach”, the sessions were a mix of groups and formal presentations, with a considerable amount of interaction amongst participants.

The development of communities of practice was focused on reports prepared by each country into their teacher education practices. Subsequent CANP programmes will build this collection of national reports. Promotion ac-

tivities had two components. The first, Gender Issues, was presented by Nouzha el Yacoubi and Daouda Sangaré. The second took the form of a tale written by Valerio Vassallo and related by him and Sidi Bekaye Sokona. This tale accompanied the exhibition “Balls and Bubbles” whose nine panels had been brought to Bamako. In addition, 15 DVDs of the film “Dimensions” was provided by Etienne Ghys and extracts were projected on the last half-day. EdiMaths was covered by Malian television and the Mali Minister of Education was present at the opening ceremony.

EdiMaths follow-up includes a regional network website, publication of the country reports and the formation of a regional community that plans to hold a second EdiMaths meeting in 2012 in Dakar.

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We are seeking sponsors for ongoing funding for future programmes in the Capacity & Networking Project. We hope that others will join the ICMI/IMU community in this major international initiative in the mathematical sciences in the developing world.

Do Theorems Admit Exceptions? Solid Findings in Mathematics Education on Empirical Proof Schemes

Education Committee of the EMS

One of the goals of teaching mathematics is to communicate the purpose and nature of mathematical proof. Jahnke (2008) pointed out that, in everyday thinking, the domain of objects to which a general statement refers is not completely and definitely determined. Thus the very notion of a “universally valid statement” is not as obvious as it might seem. The phenomenon of a statement with an indefinite domain of reference can also be found in the history of mathematics when authors speak of “theorems that admit exceptions”.

This discrepancy between everyday thinking and mathematical thinking lies at the origin of problems that many mathematics teachers encounter in their classrooms when dealing with a universal claim and its proof. The solid finding (the term “solid finding” was explained in the previous issue of this newsletter) to be discussed in this article emerged from results of many empirical studies on students’ conceptions of proof. In a simplified

formulation, the finding is that *many students provide examples when asked to prove a universal statement*. Here we elaborate on this phenomenon.

Universality refers to the fact that a mathematical claim is considered true only if it is true in all admissible cases without exception. This is contrary to what students meet in everyday life, where the “exception that confirms the rule” is pertinent. It is therefore not necessarily surprising that many students simply provide examples when asked to prove a universal mathematical claim, such as showing that the sum of any five consecutive integers is divisible by 5. Indeed, considerable evidence exists that many students rely on validation by means of one or several examples to support general statements, that this phenomenon is persistent in the sense that many students continue to do so even after explicit instruction about the nature of mathematical proof, and that the phenomenon is international and independent of the country in which the students learn

mathematics (Harel and Sowder, 2007). A student who seeks to prove a universal claim by showing that it holds in some cases is said to have an *empirical proof scheme*. The same student is also likely to expect that a statement, even if it has been ‘proved’, may still admit counterexamples. The majority of students who begin studying mathematics in high school have empirical proof schemes and many students continue to act according to empirical proof schemes for many years, often into their college mathematics years. For example, Sowder and Harel studied the understanding, production and appreciation of proof by students who had finished an undergraduate degree in mathematics. Their findings indicate the appearance of empirical proof schemes among such graduates and also how difficult it is to change these schemes through instruction. For example, one student insisted on the use of numerical examples as a way of proving the uniqueness of the inverse of a matrix.

Some mathematics teachers also hold empirical proof schemes. For example, after explicit instruction about the nature of proof and verification in mathematics, Martin and Harel (1989) presented four statements, each with a general proof and with a ‘proof’ by example to a group of about 100 pre-service elementary teachers. An example of one of the statements was: “If c is divisible by b with remainder 0 and b is divisible by a with remainder 0, then c is divisible by a with remainder 0.” Fewer than 10% of the students consistently rated all four ‘proofs by example’ as invalid. Depending on the statement, between 50% and 80% of the pre-service teachers accepted ‘proofs by example’ as valid proofs – just about the same number as accepted deductive arguments.

While the issue of empirical proof schemes has been mentioned by Polya and others, Bell (1976) may have been the first to report an empirical study about students’ proof schemes. Bell identified what he called students’ “empirical justifications” and gave illustrations. Balacheff (1987) later pointed out at least two subcategories of empirical proofs: naïve empiricism and crucial experiment. Naïve empiricism means checking specific cases, often a few cases or the ‘first few’ cases; it may include systematic checking. Crucial experiment, on the other hand, uses one supposedly ‘general’ case, say a large number; the idea behind the crucial experiment is that such a large number represents ‘any number’ and, hence, if ‘it’ works for this number then ‘it’ will work for any number.

Fischbein (1982) investigated the notion of universality. He showed that only about a third of a rather large sample of Israeli high school students reasoned according to universality. He showed that even students who claimed that a specific given statement is true, that its proof is correct and that the proof established that the statement is true in general, thought that a counterexample to the statement was possible and required more examples to increase their confidence. The issue of universality has been re-examined many times, usually with similar results. For example, when presented with an empirical argument, only 46% of a sample of German senior high school students recognised that this argument was insufficient for proving the statement. High school students in U.S. geometry classes were found to employ empirical proof schemes and did not seem to

appreciate the differences between empirical and deductive arguments. Also in the U.S., university bound students at the end of a college preparatory high school class emphasising reasoning and proof provided an example when asked to prove a simple statement from number theory.

It may be less surprising that in junior high school, about 70% of students used examples when asked to prove something (Knuth, Slaughter, Choppin and Sutherland, 2002), especially in view of the fact that a majority of teachers investigated also showed a strong use of empirical proof schemes, identifying examples as being more convincing than other proof schemes.

Empirical proof schemes may be a consequence of students’ experiences outside of mathematics classes. Mathematical thought concerning proof is different from thought in all other domains of knowledge, including the sciences as well as everyday experience; *the concept of formal proof is completely outside mainstream thinking*. Teachers of mathematics at all levels (mathematicians, mathematics educators, schoolteachers, etc.) thus require students to acquire a new, non-natural basis of belief when they ask them to prove (Fischbein, 1982). We all need to be acutely aware of this situation.

The studies mentioned above firmly establish the robustness of the phenomenon, i.e. the existence and the widespread nature of empirical proof schemes, although the following studies show that the situation is, as always in mathematics education, complex. One of the results of the London proof studies (see, for example, Healy and Hoyles, 2000) was that even for relatively simple and familiar questions the most popular approach was empirical verification, adopted by on average 34% of the students, with a much higher percentage for harder questions. This result should be considered significant since the study included a sample of 2,459 14–15 year old, high-attaining (roughly the top 25%) students from 94 classes across England (1305 girls and 1154 boys). Nevertheless, the authors concluded that even though the students appeared unable to construct completely valid proofs, many correctly incorporated some deductive reasoning into their proofs and most valued general and explanatory arguments. Additionally, these studies found that significantly more students were able to recognise a correct proof than to write one and, crucially, they made different selections depending on two criteria for choice: whether it was their own approach or to achieve the best mark. In the number/algebra questions, for best mark, formal presentation (using letters) was by far the most popular choice with empirical argument chosen infrequently. The opposite was the case for students’ own approaches, with empirical or prose-style answers much more popular than formal responses. A similar though less clear-cut pattern was reported for geometry, with ‘pragmatic’ arguments more popular for their own approach but not for achieving the best mark.

Another result, according to which many students do not grasp the universality notion, is the opacity of the notion of “logical consequence”, which is a basic ingredient in proving activities. For example, many students of different ages, when asked to check the validity of the following two “syllogistic” arguments:

- a) From the sentences “no right-angled triangle is equilateral” and “some isosceles triangles are equilateral”, it follows that “some right-angled triangles are not isosceles”;
- b) From the sentences “no dog is ruminant” and “some quadrupeds are ruminant”, it follows that “some dogs are not quadrupeds”;

answer that a) is correct while b) is not and justify their answer by observing that while the three sentences in a) are all true, the last one in b) is false (Lolli, 2005). However the two arguments are logically equivalent.

In summary, the research studies mentioned above (and it would be possible to cite many more with similar results) underline the phenomenon that students’ major approach to proving is based on empirical proof schemes. This raises a more general issue with respect to research in mathematics education (and more generally in the social sciences); are some, or even many, examples sufficient to make a finding solid? Or do we err in using an “empirical proof scheme” to establish a solid finding in mathematics education? We begin answering this question by noting that ‘argument’ in the social sciences, including mathematics education, is not equivalent to ‘proof’ in mathematics. Mathematics and mathematics education have much in common but the latter makes statements on human beings, in particular on students, teachers and teacher educators. This means that mathematics education is a complex interdisciplinary field where, in addition to mathematical issues, pedagogical, psychological, social and cultural issues also play crucial roles.

Anyway, as mathematicians and mathematics educators we might ask whether our solid finding, namely that students’ major approach to proving is based on empirical proof schemes, has a general explanation? One hypothesis is the following. Students’ specific problems with regard to proving are part of a more general challenge: to make a distinction between reasoning in mathematics and reasoning in everyday life. As mathematicians and mathematics educators, we have learned to flexibly switch between these two “worlds”. However, students, in particular young children, have little experience with mathematics as a wonderful world with its own objects and rules. They need time and support to understand this new world. This is true in particular with respect to the nature of proving which has quite different meanings in mathematics and everyday life. From this point of view, it is very well understandable that students, when entering a new field, start using the methods they have successfully used so far. Don’t we also frequently use such a strategy? Shouldn’t students’ so-called ‘misconceptions’ and ‘errors’ be regarded under this new light? Can such ‘errors’ still be regarded as individual deficiencies? Are they not, at least in part, due to an unavoidable and hard to overcome obstacle on the path of every learner of mathematics, an epistemological obstacle, an inevitable challenge that any learner has to face, namely the gap between everyday life and mathematics?

In mathematics education research we know many other manifestations of this obstacle, for example the Rosnick-Clement-phenomenon (Rosnick and Clement, 1980): when asked to algebraically express that in a

certain college, there are six times as many students as there are professors, using the variables S and P , the vast majority of students write $6S = P$ rather than $6P = S$. Regarding S and P as variables representing the numbers of students and professors, respectively, the sentence $6P = S$ represents that one should multiply the number of professors P by six in order to get the number of students S . However, students – influenced by everyday life – regard S and P as objects rather than as variables, and from that point of view writing $6S = P$ is correct since it represents that 6 students correspond to one professor. Similarly, we write 1 euro = 100 cents (not a mathematical equation!) but we would need to write the mathematical equation $100E = C$ in order to indicate that we need to multiply the number of euros by 100 in order to get the number of cents. In everyday life we rarely write $100E = C$. In mathematics classrooms, however, the students need to learn that in this particular case everyday life and mathematics have opposite ways of expressing a similar situation. This and similar situations make mathematics education challenging!

It is our task as teachers, teacher educators and mathematicians to find ways of supporting students to overcome the challenge of recognising the differences between mathematics and everyday life. The special case of proving makes students’ challenges regarding the relationship between everyday life and mathematics very visible. But it also probably shows that “errors” of individual students might have their roots in a much more general challenge. Hence we need to propose forms of proof (Dreyfus, Nardi, and Leikin, in press) that might support students in making the transition from empirical arguments to valid proofs and to investigate how such progress might be achieved. This transition includes experiencing a need for general proof, for a proof that covers all cases included in a universal statement. It also includes grasping that and why examples do not constitute proof in mathematics. The transition process also includes acquiring an ability to produce proofs that are not example-based. Research points to the transition process from empirical to conceptual proof in terms of learning how to “switch” toward the use of more formal mathematics (Leng, 2010). Students have to feel a need for general proof and make the transition to general patterns of mathematical reasoning, possibly grounded in but not relying exclusively on evidence from examples.

Concerning the need for proof, some researchers have suggested approaches that focus on how teachers can foster students’ intellectual need (Harel, 1998), whereas others have focused more on task design that generates a psychological need for proof (Dreyfus and Hadas, 1996). For example, students are likely to accept the statement that the three angle bisectors of a triangle meet in a single point as natural and hence in no need of proof or explanation. However, students may be prepared by first investigating the angle bisectors of a quadrilateral and realising that only in special cases do they intersect in a single point. Students may be further prepared by investigating possible mutual positions of three lines in a plane, seeing that they may but need not intersect in a single point. Students asked to investigate the angle bisectors of a triangle after

such preparation are less likely to expect them to intersect in a single point and are often surprised that they do intersect in a single point for any triangle whatsoever. This surprise easily leads to the question of why this happens and hence to a need for proof.

Concerning the transition to general proof, some researchers have recommended exploiting generic examples for facilitating the transition (e.g. Malek and Movshovitz-Hadar, 2011). A generic example exemplifies the general proof argument using a specific case. For example, a generic example for proving that the sum of any five consecutive integers is divisible by five might run as follows: “Let’s, for example, take $14+15+16+17+18$. The middle number is 16; the number before it, 15, is smaller than 16 by 1; the number after it, 17, is larger than 16 by 1; together these two, 15 and 17, equal 2 times 16. Similarly, the first and the last number, 14 and 18, together equal 2 times 16; hence altogether, we have 5 times 16, which is clearly divisible by 5. A similar procedure can be carried out for any five consecutive integers.”

Others have presented evidence that letting students come up with and formulate conjectures themselves may support proof production by creating a cognitive unity between conjecture and proof (Bartolini Bussi, Boero, Ferri, Garuti and Mariotti, 2007). Still others contend that carefully designing a transition from argument to proof holds some potential. This transition is particularly delicate when more sophisticated types of proofs are concerned, such as proofs by contradiction and proofs by mathematical induction. Generally, students’ mistakes in such cases are found largely to be manifestations of deficient proof schemes. It seems that pushing students’ intellectual need for proof and supporting the development of specific proof schemes in the classroom (e.g. the so-called transformational one, see Harel and Sowder, 2007) can help students in approaching more advanced forms of proof.

Finally, the method of scientific debate in the classroom has been proposed, implemented and investigated. During scientific debates, students formulate conjectures, which they consider scientifically grounded; the lecturer does not express an opinion on their correctness but manages a debate with the objective of collectively building a proof. Such debates have been organised for many years in France and their consequences have been analysed (Legrand, 2001). Compared to traditional lectures, such arguments have been found to change the attitudes of students towards mathematics, leading them to experience the need for proof.

In summary, while the findings about students’ empirical proof schemes are solid, the evidence about the transition from empirical to general proof schemes is based on limited evidence collected in suitable environments. This leaves many questions open for further research.

Authorship

Even though certain authors have taken the lead in each article of this series, all publications in the series are published by the Education Committee of the European Mathematical Society. The committee members are Ferdinando Arzarello, Tommy Dreyfus, Ghislaine Gueudet, Ce-

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Additional information

A slightly expanded version of this article with a more complete list of references may be found on the web at <http://www.euro-math-soc.eu/comm-education2.html>.

References

- Balacheff, N. (1987). Processus de preuve et situations de validation [Proof processes and situations of validation]. *Educational Studies in Mathematics*, 18, 147–176.
- Bartolini Bussi, M., Boero, P., Ferri, F., Garuti, R., and Mariotti, M. A. (2007). Approaching and developing the culture of geometry theorems in school: A theoretical framework. In P. Boero (Ed.), *Theorems in school – From History, Epistemology and Cognition to Classroom Practice* (pp. 211–218). Rotterdam, the Netherlands: Sense Publishers.
- Bell, A.W. (1976). A study of pupils’ proof-explanations in mathematical situations. *Educational Studies in Mathematics*, 7, 23–40.
- Dreyfus, T., and Hadas, N. (1996). Proof as answer to the question why. *Zentralblatt für Didaktik der Mathematik*, 28, 1–5.
- Dreyfus, T., Nardi, E., and Leikin, R. (in press). Forms of proof and proving in the classroom. In M. de Villiers and G. Hanna (Eds.), *Proof and proving in mathematics education – the 19th ICMI study*. New York, NY: Springer, NISS series, Vol. 19.
- Fischbein, E. (1982). Intuition and proof. *For the Learning of Mathematics*, 3(2), 9–18.
- Harel, G. (1998). Two dual assertions: The first on learning and the second on teaching (or vice versa). *American Mathematical Monthly*, 105, 497–507.
- Harel, G., and Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F.K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 805–842). Greenwich, CT: Information Age.
- Healy, L., and Hoyles, C. (2000). A Study of proof conceptions in algebra. *Journal for Research in Mathematics Education*, 31, 396–428.
- Jahnke, H. N. (2008). Theorems that admit exceptions, including a remark on Toulmin. *ZDM – The International Journal on Mathematics Education*, 40, 363–371.
- Knuth, E. J., Slaughter, M., Choppin, J., and Sutherland, J. (2002). Mapping the conceptual terrain of middle school students’ competencies in justifying and proving. In S. Mewborn, P. Sztajn, D. Y. White, H. G. Wiegel, R. L. Bryant and K. Nooney (Eds.), *Proceedings of the 24th Meeting for PME-NA, Vol. 4* (pp. 1693–1700). Athens, GA.
- Legrand, M. (2001). Scientific debate in mathematics courses. In D. Holton (Ed.), *The teaching and learning of mathematics at university level* (pp. 127–136). Dordrecht, the Netherlands: Kluwer.
- Leng, M. (2010). Preaxiomatic mathematical reasoning: An algebraic approach. In G. Hanna, H. N. Jahnke and H. Pulte (Eds.), *Explanation and proof in mathematics: Philosophical and educational perspectives* (47–57). New York, NJ: Springer.
- Lolli, G. (2005). *QED Fenomenologia della dimostrazione*. Torino: Boringhieri.
- Malek, A., and Movshovitz-Hadar, N. (2011). The effect of using transparent pseudo-proofs in linear algebra. *Research in Mathematics Education*, 13, 33–58.
- Martin, G., and Harel, G. (1989). Proof frames of preservice elementary teachers. *Journal for Research in Mathematics Education*, 20, 41–51.
- Rosnick, P., and Clement, J. (1980). Learning without understanding: The effect of tutoring strategies on algebra misconceptions. *The Journal of Mathematical Behavior*, 3, 3–27.