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Intuitive nonexamples: the case of triangles

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Abstract In this paper we examine the possibility of differentiating between two types of nonexamples. The first type, *intuitive nonexamples*, consists of nonexamples which are intuitively accepted as such. That is, children immediately identify them as nonexamples. The second type, *non-intuitive nonexamples*, consists of nonexamples that bear a significant similarity to valid examples of the concept, and consequently are more often mistakenly identified as examples. We describe and discuss these notions and present a study regarding kindergarten children's grasp of nonexamples of triangles.

Keywords Concept formation · Intuition · Kindergarten children · Nonexamples · Triangles

1 Introduction

Acquisition of mathematical concepts may be considered within a more general framework of concept acquisition. When discussing the general principles of concept formation, instances of a concept may also be called exemplars or examples. In mathematics, these examples are absolute, determined by the canons of mathematical correctness. Both in mathematics and in general, examples play a dual role in conceptualization. They serve as both building blocks in concept formation as well as outcomes of concept acquisition. Do nonexamples have the same duality? Are they inherent to concept formation? Do they follow from concept acquisition?

Concepts often serve as a means by which people may categorize different things, deciding whether or not something belongs to this class. In other words, one of the functions of a concept is to enable a person to identify both examples and nonexamples of the category. Thus, nonexamples follow from concept acquisition; but are they necessary for concept formation?

Concept formation is a complex process in which examples play an important role. Within cognitive psychology, several somewhat competing theories attempt to describe

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processes of categorization and of concept formation. Two major theories are the classical view and the prototypical view. According to the classical view, categories are represented by a set of defining features which are shared by all examples (Klausmeier and Sipple 1980; Smith, Shoben and Rips 1974; Smith and Medin 1981). The features of a new stimulus would then be judged against the features of a known category in order to determine if it is an example of that category. The prototypical view proposes the existence of ideal examples, called prototypes, which are often acquired first and serve as a basis for comparison when categorizing additional examples and nonexamples (Attneave 1957; Posner and Keele 1968; Reed 1972; Rosch 1973). Within mathematics education, both views are often employed when addressing the formation of geometrical concepts. Initially, the mental construct of a concept includes mostly visual images based on perceptual similarities of examples, also known as characteristic features (in line with Smith et al. 1974). This initial discrimination may lead to only partial concept acquisition. Later on, examples serve as a basis for both perceptible and nonperceptible attributes, ultimately leading to a concept based on its defining features. Such a process was described by Vinner and Hershkowitz (1980) who introduced the terms concept image and concept definition in reference to geometrical concepts. Visual representations, impressions and experiences make up the initial concept image. Formal mathematical definitions are usually added at a later stage.

Although concept formation may often begin without direct instruction, educators have long sought to use examples, as well as nonexamples, as a way to facilitate quicker and fuller concept formation (Klausmeier and Feldman 1975; McKinney, Larkins, Ford and Davis 1983). In mathematics education, the use of examples and nonexamples has largely been investigated in relation to the acquisition of geometrical concepts (Cohen and Carpenter 1980; Petty and Jansson 1987, Vinner 1991; Wilson 1986). Specifically, “nonexamples serve to clarify boundaries” of a concept (Bills, Dreyfus, Mason, Tsamir, Watson and Zaslavsky 2006, p. 127). As such, we may say that nonexamples are also inherent to concept formation. Yet, are all examples and nonexamples equally effective in the formation of a concept?

Within the set of examples, a prototypical example is intuitively accepted as representative of the concept. That is, it is accepted immediately, with confidence, and without the feeling that any kind of justification is required. Yet intuitively accepted cognitions may also cause obstacles as they have a “coercive impact on our interpretations and reasoning strategies” (Fischbein 1993, p. 233). In a similar manner, mathematics educators have come to recognize that prototypical examples are both a help and a hindrance to the formation of concepts. On the one hand, prototypical examples are easily recognizable, aiding in the initial formation of concepts (Wilson 1990). On the other hand, reasoning based on prototypes may often lead to a limited concept image. In fact, studies have shown that students tend to regard only prototypical examples as examples of the concept. Other examples, non-prototypical ones, are often regarded as nonexamples (Hershkowitz 1989; Schwarz and Hershkowitz 1999; Wilson 1990).

Some examples are intuitively accepted as representatives of a concept. Might there also be nonexamples which are intuitively accepted as such? In this paper we refer to intuitively accepted examples as intuitive examples and refer to intuitive nonexamples in the same manner. Knowing that intuitive examples have a significant impact on concept formation, it is quite possible that intuitive nonexamples have a similar impact. Or, perhaps they have a different impact. This paper is an initial investigation into the notion of intuitive nonexamples. Specifically, this study has two aims: to investigate the existence of intuitive nonexamples for triangles and to study the features that may make nonexamples intuitive.

2 Theoretical background

In this section we begin with some background theory on students' acquisition of geometrical concepts. We then review three factors which may impact on the acquisition of these concepts: naming, intuition, and prototypes.

2.1 Acquisition of geometrical concepts

Although different theories exist regarding the formation of geometrical concepts (for reviews see Battista 2007; Clements 2003; Hershkowitz 1990), in this study we use the van Hiele model (e.g., P. M. van Hiele and D. van Hiele 1958) as our basic framework and briefly describe the first three stages here. Van Hiele theorized that students' geometrical thinking progresses through a hierarchy of five levels, eventually leading up to formal deductive reasoning. In this study, our interest is focused on the beginning of this development. According to the van Hiele theory, at the most basic level, students use visual reasoning, taking in the whole shape without considering that the shape is made up of separate components. Students at this level can name shapes and distinguish between similar looking shapes. At the second level students begin to notice that different shapes have different attributes but the attributes are not perceived as being related. At the third level, relationships between attributes are perceived. At this level, definitions are meaningful but proofs are as yet not understood.

Attributes may be critical or not-critical (Hershkowitz 1989). In mathematics, critical attributes stem from the concept definition. Definitions are apt to contain only necessary and sufficient conditions required to identify an example of the concept. Other critical attributes may be reasoned out from the definition. Hence, if we define a quadrilateral as a "four sided polygon", we may then reason that the quadrilateral is a closed figure that also has four vertices and four angles. The critical attributes then include (a) closed figure, (b) four sides, (c) four vertices, (d) four angles. Non-critical attributes include the overall size of the figure (large or small) and orientation (horizontal base). One of our major aims, as educators, is to bring our students to use only critical attributes as the deciding factor in identifying examples and forming geometrical concepts. Individuals who base their reasoning on critical attributes may at the very least be operating at the second van Hiele level. If the student points out that a figure is a quadrilateral because it has four sides and therefore it also has four angles and vertices, then that child may be operating at the third van Hiele level. Hershkowitz and Vinner (1983) and Hershkowitz (1989) also found that reasoning based on critical attributes increases with age.

All examples of a concept must contain the entire set of critical attributes for that concept. On the other hand, non-critical attributes are only found in some of the concept examples. For instance, the critical attribute of equal measure when considering the four equal sides and four equal angles of the square, is a non-critical attribute when considering examples of a quadrilateral. Burger and Shaughnessy (1986) referred to the orientation of a figure as a non-critical or irrelevant attribute. Hannibal (1999), in her study of young children's understanding of shapes, found that many children reverted to the use of non-critical attributes when trying to differentiate between examples and nonexamples among similar shapes. Burger and Shaughnessy (1986) claimed that an individual's reference to non-critical attributes has an element of visual reasoning. Thus, they further claimed that a child using this reasoning may either be at van Hiele level one or at van Hiele level two, as he is pointing to a specific attribute, and not judging the figure as a whole. In fact, research has suggested that the van Hiele levels may not be discrete and that a child may display different levels of thinking for different contexts or different tasks (Burger and Shaughnessy 1986).

2.2 Naming, intuition, and prototypes

As noted above, geometrical figures may be categorized in different ways. The process of categorization by children may also be related to naming. For infants and very young children, the act of naming serves as a catalyst to form categories (Waxman 1999). In fact, categorization improves greatly when children hear a single consistent name for various examples of a category as opposed to hearing different names for the different examples (Waxman and Braun 2005). Interestingly, Markman (1989) proposed that when children hear a new name for an object, they assume it refers to a whole object and not to its parts. This coincides with the van Hiele levels in which children first take the whole shape into consideration without regarding its components. Studies have also shown that children assume a given object will have one and only one name (e.g. Markman and Watchtel 1988). This assumption may cause difficulties in accepting the hierarchal structure of geometric figures where a square is also a rectangle and a quadrilateral.

The notion of intuition has also been shown to play an essential role in the mathematical thinking processes of students (Fischbein 1987). Intuitive knowledge is both self-evident and immediate and is often derived from practical experience. As such it does not always promote the logical and deductive reasoning necessary for developing geometrical concepts. Fischbein (1993) considered the figural concepts an especially interesting situation where intuitive and formal aspects interact. The image of the figure promotes an immediate intuitive response. Yet, geometrical concepts are abstract ideas derived from formal definitions. “Very often the intuitive representation is stronger and tends to annihilate the formal conception.” (Fischbein 1987, p. 205). In the case of examples and non-examples, intuitively accepted examples and non-examples would be those figures which children immediately identify as such, feeling no need to justify their claims.

Prototypes also play an important role in the formation of concepts. In relation to geometrical concept formation and tasks, Hershkowitz (1989) claimed that in addition to the necessary and sufficient (critical) attributes that all examples share, prototypical examples of a shape have special (non-critical) attributes “which are dominant and draw our attention” (p. 73). The prototypical examples often have the longest list of attributes. Initially, children’s concept images consist of prototypical examples. In drawing tasks, children most often draw a prototypical example. Hershkowitz (1989) found that even when an invented concept is introduced solely by a verbal definition, a prototypical shape emerges from students’ drawings. Smith et al. (1974) argued that some examples, namely prototypical examples, are rapidly identifiable as an example of the category, whereas other examples may take longer to identify. They also hinted at the possibility that some nonexamples are so dissimilar that they are rapidly identified as being nonexamples of the category. Clements, Swaminathan, Hannibal and Sarama (1999) suggest that different shapes may have different numbers of prototypes. They reported that the circle and square have fewer prototypes than rectangles and triangles. Some studies have suggested that overexposure to prototypes may impede the growth of fuller concept acquisition. For example, Kellogg (1980) suggested that prototypes are formed when certain non-critical attributes of a shape appear frequently in examples and students begin to associate these non-critical attributes with examples of the shape. Wilson (1986) advocated the use of nonexamples in order to lessen the effect of prototypes. By exposing students to nonexamples with the same non-critical attributes, students may begin to differentiate between critical and non-critical attributes.

To summarize, the acquisition of geometrical concepts is a complex process which includes both visual and attributional reasoning. Naming, intuition, and prototypes play a major role in geometric conceptualization. Past research has focused on differentiating between different types of examples, including intuitively accepted prototypical examples. This research focuses on nonexamples, specifically the nature of nonexamples which are intuitively identified as such.

3 Methodology

3.1 Preliminary study

In order to gain insight into the types of nonexamples that should be used in this study, 28 adults with at least a first degree in science, mathematics, or engineering were asked to give an example of something that is not a triangle. They were then each asked to give another example of “something that is not a triangle". The immediate first response of all the adults was a circle. Their second example was a square (24 adults) or a rectangle (4 adults). At this point, 75 prospective elementary school teachers were asked to give an example of something that is not a triangle. Their immediate responses were a circle (58 prospective teachers), a square (11), a circle and a square (4), a circle and a rectangle (1), and a “triangle” with curved sides. Finally, 22 kindergarten children, four and five years old, were asked to give an example of something that is not a triangle. The immediate first response of 18 children was a circle. The four others said a square. When asked to give another example, those that had first said a circle responded with a square and those that had first responded with a square, said a circle. There are two important results of this preliminary study. First, every participant replied by naming a two-dimensional geometric figure. Thus, the sample space (two-dimensional geometric figures) was implicitly understood. Second, 83% of the participants immediately named the circle as an example of something that is not a triangle. Almost all of those that did not name a circle immediately named a square and where the circle was the first non-triangle named, the square was named second.

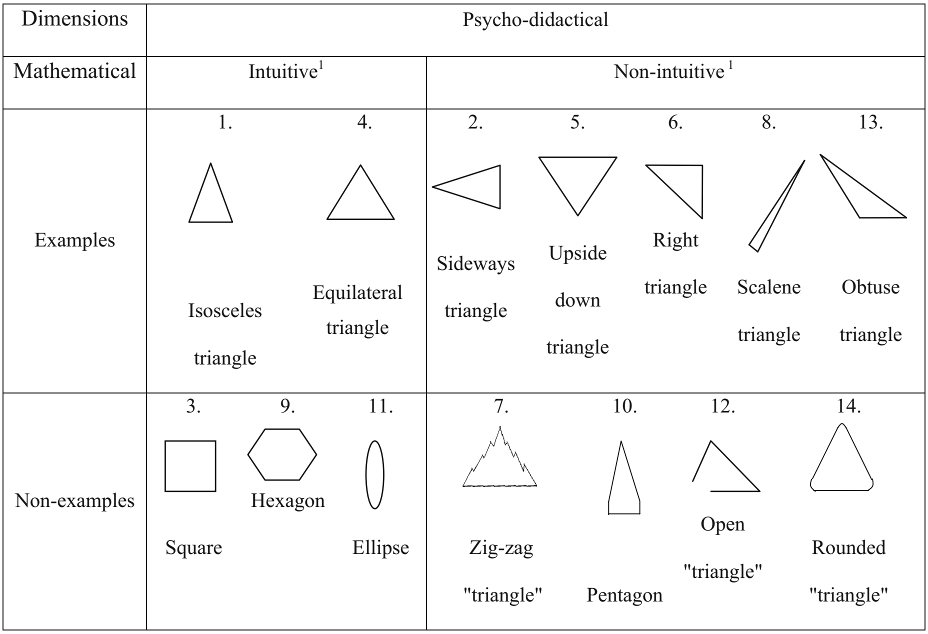
3.2 Participants

The participants in this study were 65 children between the ages of five to six years old. These children learned in four different state kindergartens located in two middle socio-economic neighborhoods.

3.3 Tools and procedure

Children were interviewed individually, in a quiet corner of the kindergarten classroom. Fourteen different figures were used for this study, each figure printed on a separate card. The figures and the order in which they were given are shown in Fig. 1. The order was the same for each child. After presenting each card, in the same order to each child, two interview questions were asked: Is this a triangle? Why? The first question ascertained if the child identified examples and nonexamples of a triangle. The second question allowed us to study the child’s reasoning about identification of a figure. The interviewer listened to and wrote down what the child said and did, including gestures. Immediacy of responses was noted.

Seven examples and seven non-examples were included in the figures. Following Hershkowitz (1990) the equilateral and isosceles triangles (figures *1* and *4*) were considered



[1] Based on research results

Fig. 1 Figures presented to the kindergarten children

to be intuitive examples. The other five examples (figures *2, 5, 6, 8*, and *13*) were all considered non-intuitive examples. For example, Shaughnessy and Burger (1985) found that young children did not identify as a triangle a long and narrow triangle, such as the scalene triangle in figure *8*, even when they admitted that the figure had three points and lines.

The non-examples were all two-dimensional shapes gathered from three categories: prototypical geometrical shapes (other than triangles), non-prototypical geometrical shapes (other than triangles), and “almost triangles". Preliminary studies had shown that when asked to give an example of something that is not a triangle, many people will respond with either a square or a circle. It was decided to include in the set of non-triangles one of these figures (the square – figure *3*), an additional regular polygon (the hexagon – figure *9*) as well as a different closed prototypical non-polygon shape (the ellipse – figure *11*). In the second category, non-prototypical geometrical shapes, is the pentagon (figure *10*). The pentagon used in this study is non-prototypical of pentagons. However, it was positioned with a horizontal base, in a similar manner as the prototypical triangle, and was elongated in such a manner as to visually suggest a triangle. The third category, “almost triangles” consisted of shapes that have one or more attributes missing but otherwise share most of the attributes of the prototypical triangle. In this category are the zig-zag “triangle", open “triangle", and rounded “triangle” (figures *7, 12*, and *14* respectively). The open “triangle” is missing the critical attribute of being a closed figure. The zig-zag “triangle” has jagged sides. The rounded “triangle” is missing vertices. On the other hand, all have horizontal bases and all have the illusion of threeness. Some of these figures have been investigated in other studies. For example, Hasegawa (1997) found that the rounded “triangle” is often identified as a triangle. Regarding the open “triangle", some studies have shown that “openness” is regarded by many students to disqualify a figure from being a polygon

(Hershkowitz and Vinner 1983) while others have found that it is not necessarily a disqualifier (Rosch and Mervis 1975). The zig-zag "triangle" was a figure created for this study. If one zooms in on the non-horizontal sides, then the correct definition of this figure would be a 15-sided polygon, thus losing the critical attribute of threeness. However, zooming out, the overall picture one perceives is that of a three-sided figure of which two sides are jagged, losing the critical attribute of straightness. This figure enabled us to study the child's ability to home in on the details as opposed to the overall shape of the figure. Taken all together, the group of non-triangles afforded us the opportunity to begin an investigation into what makes a non-triangle intuitively accepted as such. Is it the overall shape of the figure? Is it the ability to name the figure? Is it the number of missing critical attributes? Is it the particular missing attribute?

3.4 Analyzing the data

Two sets of data were analyzed, corresponding to the two interview questions. The first set of data consisted of children's responses to the question of identification. Being that one of the characteristics of intuition is immediacy (Fischbein 1987) we were interested in responses that were correct immediately as opposed to a child who had to pause or debate between including the figure in the set of triangles or not. Four scores were given to this set of data: correct immediately, correct but not immediately, incorrect immediately, and incorrect but not immediately.

The second set of data resulted from children's reasoning about identification of a figure (see Table 1). Using the van Hiele levels of geometrical thought, children's reasoning was first categorized into visual reasoning and reasoning based on the figure's attributes. Data within each category was then analyzed using a finer grain. In the category of visual reasoning, two sub-categories emerged. In the first sub-category were responses based on appearance alone where the figure was perceived as a whole. An example of such reasoning was one child, K14[1], who claimed that the hexagon was not a triangle because, "You don't see the shape." The second sub-category consisted of the child naming the figure. Some children named the figures using geometrical shape names, while others used non-geometrical names. For example K36 claimed that the pentagon was not a triangle because "it's similar to a tent." Clements et al. (1999) suggested that naming figures as shapes or objects is also a type of visual reasoning. Although we agree that this type of reasoning is mostly visual, we view this level of reasoning as different from the previous category where the child merely states that he "doesn't see" a triangle.

The second level of van Hiele thought is reasoning based on attributes. As discussed in the background, attributes may be further divided into critical and non-critical attributes. For the purpose of this study, we used a non-minimalist definition of a triangle: A triangle is a closed figure with three vertices and three straight sides. Although minimalism is considered a basic principle of mathematical definitions, there are precedents (namely, the definition of congruent triangles) where a non-minimal definition may be psychologically and didactically preferred (Linchevsky, Vinner and Karsenty 1992). We realize that in a closed figure, three vertices imply three sides. However, some children may not necessarily see this implication so a differentiation was made. Furthermore, the term "straight" is superfluous when one understands that a side is indeed straight. However, knowing that young children may not realize that a side must necessarily be straight, we included this

[1] Kindergarten children were labeled K1-K65.

Table 1 Coding reasons after identifying a figure

Category	Reasons
Purely visual reference to the whole figure	"It looks (doesn't look) like a triangle." "You see (don't see) the shape." Traces the figure without saying a word.
Naming	"It's a rhombus (or some other geometric shape – correct or incorrect)." "It's a bonfire (names an object)."
Reference to non-critical attributes	"Because this (points to a particular side) is too small (short, big, long)." "It's (referring to the figure) too thin (fat, long, sharp)"
Reference to critical attributes	"It has three (four, five, many, none) sides (lines, points, corners)." "It has to be closed." "It has three rounded points."

attribute. According to our working definition of a triangle there are four critical attributes: (a) closed figure, (b) three, (c) vertices, (d) straight sides.

Non-critical attributes are "usually attributes of a prototypical example only." (Hershkowitz 1989, p. 69). These attributes might refer to the length of the sides, the measurement of the angles, or the orientation of the figure. Although reasoning based on non-critical attributes should fall under the second van-Hiele level of attribute reasoning, it might also be considered partly visual. Comparing a figure to the prototypical examples is what Hershkowitz (1990) called prototypical judgment. This may be partly a visual judgment as the "prototype's irrelevant attributes usually have strong visual characteristics." (p. 83). Taking all of this into account we suggest that reasoning based on non-critical attributes may serve as a bridge between the first and second van Hiele levels of thought. Our third category, therefore, was reasoning based on non-critical attributes. For example, when discussing the open triangle, K49 claimed that it was not a triangle because "it has here (pointing to the right side) a long [line] and here (pointing to the left side) it's short." The fourth category was reasoning based on critical attributes. Some children correctly used the critical attributes by counting sides or vertices, for example. Others referred to critical attributes but applied them incorrectly. For example, one child (K63) looking at the rounded triangle said, "It's a triangle because there are three corners." This response indicates that the child is aware that a triangle must have three vertices. However, his conception of a vertex is that of a corner which are not necessarily one and the same.

Responses could be verbal or could be a gesture, such as tracing the whole figure with one's index finger or pointing to a specific area of the figure. Table 1 lists common examples of children's reasoning and their categorization. Children who gave more than one reason in two different categories were given more than one code, in accordance with the appropriate categories.

4 Results

In order to investigate the existence of intuitive nonexamples, we begin by reviewing children's identification of both triangles and non-triangles. This affords us a general background of what children consider to be an example or non-example and which examples and nonexamples may be considered intuitive. We then focus on the non-triangles and review the basis for children's reasoning regarding these figures. As described in the previous section, a differentiation is made between visual and attribute reasoning and between different types of visual and attribute reasoning. Specifically, we analyze the different features of the nonexamples and the different types of reasoning which were associated with these features. This allows us to investigate the features that may make nonexamples intuitive.

4.1 Identification

Over 90% of the children correctly and immediately identified the intuitive triangles as examples whereas no more than half of the children correctly identified the non-intuitive triangles. It should be noted that the results for the triangles were on par with results of previous studies (e.g., Clements et al. 1999).

In general, more children correctly identified the non-triangles than the triangles. Concerning the non-triangles, what stands out most are the non-triangles that 100% of the kindergarten children identified as such. These were the square, hexagon, and ellipse. In other words, all of the prototypical geometrical shapes were easily and immediately identified by most children as non-triangles. Approximately 80% of the children correctly identified the non-prototypical pentagon, the zig-zag "triangle", and open "triangle" as non-triangles. Finally, only three children (5%) identified the rounded "triangle" as a non-triangle.

4.2 Reasons

As mentioned in the previous section, children's reasons for identifying a figure as a non-triangle were categorized into four types of reasoning. Some children gave several reasons classified in different categories. There were 35 instances of responses which could not be categorized because either the child did not give a reason or because the given reason was incomprehensible. Results are summarized in Table 2. It should be noted that all four categories led to both correct and incorrect identification of non-examples. In this section,

Table 2 Frequency of reasons accompanying non-triangles

Figure		Type of Reasoning							
		Visual Reasoning				Attribute Reasoning			
		Whole shape		Naming		Non-critical attributes		Critical attributes	
Identification:		Correct	Incorrect	Correct	Incorrect	Correct	Incorrect	Correct	Incorrect
Intuitive non-triangles	Square	14	–	34	–	4	–	22	–
	Hexagon	14	–	23	–	4	–	21	–
	Ellipse	9	–	43	–	–	–	12	–
Non-intuitive non-triangles	Pentagon	11	4	8	2	16	–	10	6
	Zig-zag	7	3	18	4	1	–	28	5
	Open	17	7	–	1	–	1	37	5
	Rounded	–	23	–	1	1	4	2	33

we first look at some general trends and then look separately at the three categories of non-examples presented in this study.

4.2.1 General trends

Taking a step backward, we first look at the combined results of the first two categories representing visual reasoning, versus the combined results of the second two categories, representing attribute reasoning. From this point of view, more reasons were based on visual cues than on specific attributes. However, when reverting back to the four separate categories, we note that most reasons were based on critical attributes, followed by, in decreasing preference, naming the figure, whole shape reasoning, and reasoning based on non-critical attributes. Finally, we note that the square was the figure for which children most often gave more than one reason. A total of 74 reasons were recorded for this figure given by 65 participants. Fifty-seven reasons were recorded for the pentagon, the figure for which the least amount of reasons was recorded.

4.2.2 The square, hexagon, and ellipse

Regarding the prototypical shapes (square, ellipse, and hexagon) a few interesting results were observed. Unlike the other non-triangles, reasoning regarding these shapes was mostly based on ability to name the shape. Regarding the square, a little more than half of the responses consisted of simply identifying this figure correctly as a square, which apparently was enough to exclude it from the category of triangles. Regarding the ellipse, approximately half of the responses consisted of naming this figure an ellipse or circle. Fifteen per cent of the responses referred to some object, such as a mirror or an egg. The hexagon was an exception. Only 24% of the responses included naming the hexagon as some geometrical shape, including naming it as a rectangle or trapezoid and not necessarily a hexagon. Even fewer (13% of the responses) referred to it as some object. Perhaps this drop in visual reasoning was due to the hexagon being less familiar to children of this age group. Yet, the decrease in this type of reasoning for the hexagon was not combined with an increase in other types of reasoning.

4.2.3 The non-prototypical pentagon

Looking at the non-prototypical shape of the pentagon, a different exception to a general trend was observed. Whereas for the other non-triangles, no more than 6% of the reasons were based on non-critical attributes, when it came to the pentagon, 28% of the responses consisted of this type of reasoning. Furthermore, this type of reasoning consistently went along with correct identification of this figure as a non-triangle. Recall that the pentagon was a non-prototypical pentagon and was actually constructed to be somewhat similar to a triangle. Typically, children who used this type of reasoning commented on the figure's thinness or stretched out look. It is equally important to note that reasoning based on critical attributes had a relatively small success rate for the pentagon. Furthermore, the critical attributes children referred to were sometimes erroneous. For example, one child (K19) who correctly identified the pentagon as a non-triangle claimed "the sides are crooked." In other words, this child knew that a triangle must have three straight sides. Of the children who used this reasoning when incorrectly identifying the pentagon as a triangle, many claimed that the pentagon had three points or three sides. It is not clear whether these children saw extra sides and points but ignored them or if they did not notice the extras. Perhaps some children thought that two of the sides were not straight but straightness for these children was not critical.

4.2.4 The "almost triangles"

In the group of "almost triangles", a few notable results were observed. First, we focus on the open and rounded "triangles", and note that only one child gave a name for each of these figures. On the other hand, more responses (over 35%) consisted of visual reasoning based on the whole figure for these non-triangles than for any of the other non-triangles. This is not surprising. Recall that all three figures in this group were constructed to look like triangles and not some other recognizable shape. This type of reasoning led to correct or incorrect identification depending on whether the child thought that it looked like a triangle, or not. The exception in the group was the zig-zag "triangle". This figure stimulated the children's imagination. More responses (33%) consisted of naming this figure as some object (a bonfire, mountain, or thorn bush) than was done for any of the other figures in this study. This kind of reasoning was usually accompanied by a correct identification.

Another important result in the sub-group of "almost triangles" was that considerably more reasons were based on critical attributes when identifying these figures than for the other non-triangles. This result was especially notable for the open "triangle", where 62% of the responses included this type of reasoning. Yet, this reasoning was not always accompanied by a correct identification. Some children simply stated that "it's still a triangle, even if it's open." The actual word "open" was only mentioned in 28% of the responses. Interestingly, 20% of the reasons referred to the amount of vertices being less than three. This second comment actually shows that some children knew that a vertex must be the connection of two segments and not just the end point of one segment.

Regarding the zig-zag "triangle", 85% of the responses which referred to critical attributes were associated with correct identification of this figure as a non-triangle. One child, K48, referred to the "thorns" on the sides. It is not clear if this child was referring to sides that were not straight or to excess points. Other children were more precise claiming "there are lots of corners and points." On the other hand, using critical attributes did not guarantee correct identification. One child (K56) claimed that it was a triangle because "it has three corners." It seems that this child zoomed out, looked at the overall shape, and focused on the vertices that were in the prototypical position. Yet, he used a critical attribute to make his judgment.

Regarding the rounded "triangle", 42% of the critical attribute reasons focused on the three "sides" of the "triangle". These were consistently associated with an incorrect identification. The rest focused on three "points" or "corners". While most children did not comment on the roundness, four children pointed to the three rounded corners and claimed, "it has three corners even though it's rounded." These children did not regard roundness as disqualifying the figure from being a triangle.

When considering the way the group of "almost triangles" was constructed, the fact that more children based their reasoning on critical attributes for this group than for the other two groups is especially interesting. The zig-zag "triangle" was missing one, possibly two critical attributes, depending on the focus of the child. Zooming in, the zig-zag "triangle" had more than three vertices and sides. Zooming out, the zig-zag "triangle" had two "sides" that were not straight. The rounded "triangle" was missing vertices. Yet, more children focused on the critical attribute of openness than on the other missing critical attributes. This raises two questions: Are all critical attributes equal in the eyes of children? Is it more noticeable when an attribute is missing than when it is there but in a deformed manner?

4.2.5 Consistency

We end this section with a word on consistency. Results indicated that most children were not consistent in the types of reasoning they exhibited regarding the non-triangles. Illustrating this tendency was one child (K11), who correctly counted six points on the hexagon, identifying it as a non-triangle, but did not count the vertices of the pentagon. Instead, he claimed that the pentagon was not a triangle because "it's a little squashed." Yet, he claimed that the zig-zag "triangle" was a triangle "even though it has a lot of points." A triangle has three vertices. This one critical attribute was used by the same child to correctly identify one figure as a non-triangle but was then ignored to incorrectly identify the zig-zag "triangle" as a triangle. And instead of using it for the pentagon, which had five vertices, he used a non-critical attribute. This child's reasoning illustrates the complexities involved when reasoning about non-examples.

Which returns us to our original questions: Are some nonexamples intuitively identified as such? And if so, what are the features that contribute to its being intuitively accepted as a nonexample?

5 Discussion

In this section we summarize our findings regarding non-triangles, including the notion and features of intuitive non-triangles. We then discuss some implications for instruction.

The square, hexagon, and ellipse were immediately identified as nonexamples by all the children. Furthermore, fewer children used attribute reasoning for these figures than for the other non-triangles. In other words, the children did not feel the necessity to go beyond the whole image in order to justify their identification. Immediacy and self-evidence imply that these three nonexamples were intuitively accepted as such by the children.

The child's ability to name a figure seems to have played an important role in the intuitive identification of non-triangles. It is not surprising that the children could correctly name a square thereby intuitively identifying it as a non-triangle. This goes along with Markman's (1989) theory of mutual exclusivity. If the child can name the figure a square, then, for him, that figure cannot be a triangle. Yet, only half of the children named the ellipse as an ellipse or a circle and even fewer children correctly named the hexagon. Instead they gave these figures a different name, sometimes naming a different geometric figure and sometimes naming an imaginative object. This raises an interesting question. Did the children first recognize these figures and their names, thereby identifying them as non-triangles, or did they first identify these figures as non-triangles, and as a result, feel the need to name them? In other words, it may be that the relationship between intuitive non-examples and naming could go in both directions. If the child can name the (non-triangle) figure, then intuitively it cannot be a triangle. If the child intuitively recognizes the figure as a non-triangle, then it must be some other figure for which a name must exist and be given.

In geometry, a nonexample of a concept is an instance which is missing at least one critical attribute of the concept being considered. The circle, from our preliminary study, and the ellipse from our current study, each have a relatively long list of missing critical attributes. Other than being closed figures, each has no vertices, no straight sides, and no angles. Perhaps the more critical attributes a nonexample is missing, the more likely it will be intuitively accepted as a nonexample.

It could also be that children consider some critical attributes to be more important than others. The square and hexagon, intuitively accepted as nonexamples, were each only missing the critical attribute of threeness. On the other hand, the pentagon, as well as the "almost triangles" all gave the illusion of threeness. Perhaps the association between a triangle and the attribute of threeness is stronger than the necessity for it to be closed or for its vertices to be pointy. Furthermore, it might be argued that the bond between a triangle and its attribute of threeness is also expressed in the name itself, which in many languages, including Hebrew, stems from the root three. So, if a child perceives threeness in a shape, then the child sees a triangle. Conversely, a shape which is missing threeness cannot be a triangle.

This illusion of threeness, rather than the actuality of threeness, is reminiscent of a prototype (Hershkowitz 1990). Prototypical examples appear to play an important role in geometric concept acquisition. If children intuitively accept prototypical examples then may we regard intuitively accepted nonexamples as prototypical nonexamples? More specifically, do prototypical nonexamples exist for triangles? Rosch (1973) claimed that a prototype is the example most often chosen as representative of a category. As a result, when asked to give examples of a category, people name prototypical members first. When considering the non-triangles, our preliminary studies showed that the circle and square are the first examples to be recalled. Our current study showed that the hexagon and ellipse could also be candidates of prototypical examples of non-triangles, or prototypical non-triangles.

Because there are various ways in which a shape can be different from a triangle, it may seem inappropriate to discuss prototypes for nonexamples. On the other hand, although an ideal non-triangle may not exist, there do seem to be better and worse examples of non-triangles. From an educational standpoint, it is important to understand the ramifications of possible prototypical nonexamples. These nonexamples, like prototypical examples, may have a significant impact on the child's acquisition of concepts. Further investigation is needed in order to discern if intuitively accepted nonexamples have this impact on children's concept acquisition.

It is often argued that instruction of geometrical concepts should include more than mere exposure to prototypical examples of the concept (Clements et al. 1999; Hershkowitz 1989). Similarly, we suggest that geometry instruction include exposure to a variety of nonexamples, and not merely intuitive nonexamples. In line with Watson and Mason (2005), who coined the term "personal example space" and observed that very often learners have a very limited collection of examples in mind, we argue that a significant aim of learning mathematics is extending and enriching the space of examples to which one has access. The study of nonexamples, including the notion of intuitive non-examples and possibly the notion of prototypical nonexamples could significantly contribute to achieving this aim.

The use of nonexamples has been shown to encourage students to reason out loud, providing opportunities for teachers to examine students' thinking (Clements et al. 1999). In geometry, as well as in other mathematical domains, our goal is to encourage students to construct concept images that are consistent with the concept definitions, and to promote the use of a definition as the decisive criterion for determining if an object is an example of a given concept. In geometry, specifically, we allow that visual judgment may be a necessary first level, but that analytical judgment based on critical attributes should follow. In this study we found that not all nonexamples promote the same type of reasoning. The square, hexagon, and ellipse seemed to have encouraged more visual reasoning than the other nonexamples. This intimates that intuitive nonexamples, like intuitive examples, may encourage visual rather than analytical thinking. It then becomes the teacher's role to point out the critical attributes. The same may be said for the pentagon, which seemed to have promoted reasoning based on non-critical attributes. On the other hand, it seems that the group of "almost triangles" encouraged children to use reasoning based on critical attributes.

As noted in the beginning of this paper, nonexamples play a dual role in conceptualization. As building blocks in concept formation, it is important to differentiate between intuitive and non-intuitive nonexamples and understand how they may impact on children's thinking. However, nonexamples are also outcomes of concept acquisition. As such, having the students identify both intuitive and non-intuitive nonexamples may reflect on the conceptual level the student has attained, affording the teacher an opportunity to gain important knowledge of her students. Both of these roles need to be considered when organizing and presenting intuitive and non-intuitive nonexamples during instruction.

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