How to teach operations with integers

including printable summary fact sheets for download

Addition

1. **Number line.** Addition of integers on a number line is presented as a movement of so many units either right or left. The first number in the expression is your "starting point". If you add a positive integer, you move that many units right. If you add a negative integer, you move that many units left.

   For example, 5 + (-6) means you start at 5, and you move 6 units to the left. -9 + 5 means you start at -9, and move 5 units to the right.

   This idea is usually relatively simple for students to grasp.

2. **Counters.** These are represented as little balls with + or - sign drawn inside them, or something similar. For example:

+ + + + + − − −	− − − − − − − − + + +
This represents 5 + (-3).	This represents (-8) + 3.
Each plus-minus pair cancels, and so the answer is positive 2.	Each plus-minus pair cancels, and so the answer is -5.

Subtraction

You have several options how to present subtraction of integers. Personally, I think of situations where we subtract a *positive integer* in terms of the number line, and the situations where we subtract a *negative integer* ("the double negative"), I change those to additions.

One is the familiar number line. Try split it to two cases:

1. **Number line.** Here, 2 − 5 would mean that you start at 2, and you move 5 units to the left, ending at -3. This is identical to interpreting the addition 2 + (-5) on the number line.

   Similarly, -4 − 3 would mean that you start at -4, and you move 3 units to the left, ending at -7. This is identical to interpreting the addition -4 + (-3) on the number line.

Subtracting a *negative* integer using number line movements is a bit trickier. Problem such as -4 − (-8) would mean that you start at -4, you get ready to move 8 units to the left (the "minus sign"), but the second minus sign reverses your direction, and you go 8 units to the right instead, ending at 4.

2. **Patterns** can be used to justify the common rules for subtracting integers. First, consider subtracting a positive integer. For example, consider 2 - 5. Do a little pattern for the student to solve, and observe what happens with the answers:

```
3.    3 − 1 =
4.    3 − 2 =
5.    3 − 3 =
6.    3 − 4 =
7.    3 − 5 =
   3 − 6 =
```

Also here you can use the number line. For example, 5 - 8. Place your finger at 5, and show or draw an arrow that is 8 units long towards the left. You will 'end up' at (-3).

Then do the same when your starting point is a negative number, such as (-4) - 5. Start at (-4) and move 5 units to the left.

Even when subtracting from a negative integer you can use a pattern, and ask the student to observe the answers, and then continue the pattern:

```
(-4) + 2 =
(-4) + 1 =
(-4) + 0 =
(-4) − 1 =
(-4) − 2 =
etc.
```

Also use temperature dropping examples:
5 - 9 means temperature is 5 degrees and drops 9 degrees.
(-4) - 8 means temperature is -4 now and drops 8 degrees.

Pattern to justify the rule for subtracting a negative number. This is the case with a problem such as 7 − (-2) or (-4) − (-3).

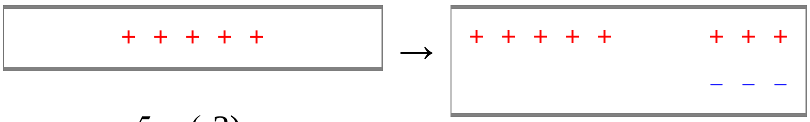
Observe the pattern and see what happens:

```
3 − 3 =
3 − 2 =
3 − 1 =
3 − 0 =
3 − (-1) =
3 − (-2) =
3 − (-3) =
3 − (-4) =
```

Students are led to discover the shortcut that *two negatives turns into a positive!*.

8. **Counters.** These are trickier to use with subtraction, but the basic idea is to interpret subtraction as "taking away". For example, for (-4) − (-2), you start out with 4 negative counters, and you take away two negative counters. So you are left with 2 negative counters.

   In other situations, you may not initially have the counters that you are supposed to take away. For example, in 5 − (-3), you start out with 5 positive counters, but you are supposed to take away 3 negative counters. How to do that? The trick is to first *add* enough pairs of negative-positive counters to the situation - this amounts to adding zero, so it is alright. Then you can take away what you need.

   5 − (-3).

   We cannot take away three negative counters, so we'll add three negative-positive pairs (which amounts to adding zero).

   Now, taking away three negatives leaves +8.

9. **Difference.** Remind the students that 5 − 2 denotes the difference of 5 and 2, which is 3. You can think of the difference as *the distance* between the two numbers on the number line. However, you need to write the greater number first! If we wrote 2 − 5 instead, we'd have to take the distance as a negative number.

   Using this model, (-2) − (-9) would mean the distance between -2 and -9, which is 7. However, (-9) − (-2) would be -7, because the numbers wouldn't be in the order of having the greater number first. Similarly, 4 − (-2) would be 6 since that is the distance between 4 and -2. -6 − (-3) would have the numbers in the "wrong" order, so we'd take their distance as a negative number and the answer would be -3.

The video below shows how to use THREE of the different models for subtraction of integers: 1) the number line model, 2) Concept of difference, and 3) counters.

[Subtracting Integers]

Multiplication

Multiplying with negative numbers is EVENTUALLY quickest to do by just memorizing the little rules:

negative x negative is positive
positive x positive is positive

negative x positive is negative
positive x negative is negative.

In other words, if the two integers have a different sign, then the product is negative, and otherwise it's positive.

But if your student or you would like to know a little bit as to WHY it all works that way, use this:

1. 3 × (-8) or when you have positive × negative:

   This can be written as repeated addition:
   (-8) + (-8) + (-8) = -24

2. (-5) × 4 or negative times positive.

   By the fact that multiplication is commutative, you can turn this around and then by 1) above, it is negative:

   (-5) × 4 = 4 × (-5) =
   (-5) + (-5) + (-5) + (-5) = -20.

3. Negative times negative. Make a pattern:

   (-3) x 3 =
   (-3) x 2 =
   (-3) x 1 =
   (-3) x 0 =
   (-3) x (-1) =
   (-3) x (-2) =
   (-3) x (-3) =
   (-3) x (-4) =

   and observe how the products continually increase by 3 in each step.

   Another justification for this rule can be seen using distributive property.

   Distributive property of arithmetic states that a(b + c) = ab + ac.

   So, if a = (-1), b = 3, and c = (-3), it should still hold:

   $$(-1)(3 + (-3)) = (-1)(3) + (-1)(-3)$$

   Now, since 3 + (-3) is zero, the whole left side is zero.

   So (-1)(3) + (-1)(-3) must be zero as well.

(-1)(3) is -3. So it follows that (-1)(-3) has to be opposite of -3, or 3.

This last part might be too difficult for 6-7th graders to grasp.

But they don't have to grasp it all; you can say that sometimes we have to just follow the rules and understand the "why" fully later. They can probably understand it partially now.

The negative x negative makes positive rule has to do with the fact that IF we made it to he positive, then all these neat rules/properties of arithmetic wouldn't hold for negative numbers... but since we want them to hold, since we DO want mathematics to be a very consistent system, then we make negative x negative to be positive.

Division of integers

Division follows because it's the opposite operation of multiplication:

What is (-21) ÷ (-7) ? I call the answer A.

(-21) ÷ (-7) = A.

It follows that A × (-7) = (-21)

Knowing the multiplication rules, the only number that fits A is 3.

And so on. Just make a similar case for (-21) ÷ 7 and 21 ÷ (-7).

(In reality, mathematicians would not use specific numbers like 21 and 7 but just variables; I wrote this with specific numbers to make it easier to grasp the argument, plus this is the way you'd probably explain it to a 6th or 7th grader.)

Of course with division too the student will just memorize the little rules and use those in practical computations.

But studying the logic behind all this is very enlightening.