Getting Started

Work with either the basic commercial set (small white & red squares, large white & green squares, and long white & green rectangles) or cutouts from one of the attached sheets. The commercial sets are nice and sturdy, but tempt students to negative numbers too quickly and cost too much to give away a set to every kid. The sheets are not as sturdy, but can be made available to kids to take home and delay the idea of negatives until the teacher is ready to cope with it. You may also have a commercial set with other colours for a second variable, and some more squarish rectangles. Hide them for the moment. The third sheet in the cutout set provided fulfills the same functions, so ignore it for the moment, too.

Naming

I call the small squares "nameless ones." This captures two important ideas: They aren't named when we talk about polynomials, and their value is 1. The reason for the name isn't something you'd tell kids on the first day, but having reasonable names for things makes the going easier later on.

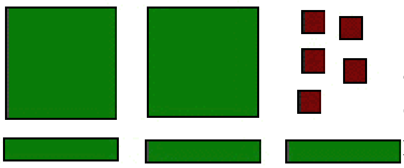
Nameless ones behave like the bingo chips or whatever from learning integers, and like the bears the kids counted in kindergarten. One of them stands for 1, and five of them stands for 5. If you are using the cutout ones, they are one square centimeter, and you can make a connection to measurement if you want.

What do you want to call the other pieces? Settle on names within groups, and within the class. Naming conventions are important to mathematical communities. You might find you have pieces which look similar to another group's but not exactly the same (rectangles that are longer than yours for example). It's OK to give these the same names because they are used in the same way.

Some names which students have proposed in the past are: Longs & Big Squares, Rectangles & Rectangle squares, and X and X^2. The point to emphasize with names is not that there's a correct name for each piece, but that we need names to be able to talk about them. The mathematical concept here is that a variable is a *name* for something else (usually an unknown or variable quantity).

Collections, part 1

Make a collection of pieces, using all three kinds, and write down its name. Settle on a convention for writing collections which shows that all the pieces are added together in one collection. Abbreviations might help. You could call this collection:



2BS + 3R + 5 (for two big squares, three rectangles and five nameless ones)
or $2R^2 + 3R + 5$ (for two rectangle squares, three rectangles, and five nameless ones)
or $2x^2 + 3x + 5$ (for two X^2, three X, and five ones)

Here you want to try to get the names sorted out, so that groups can describe their collections. Have the students draw their collections, and write the names of the collections with symbols.

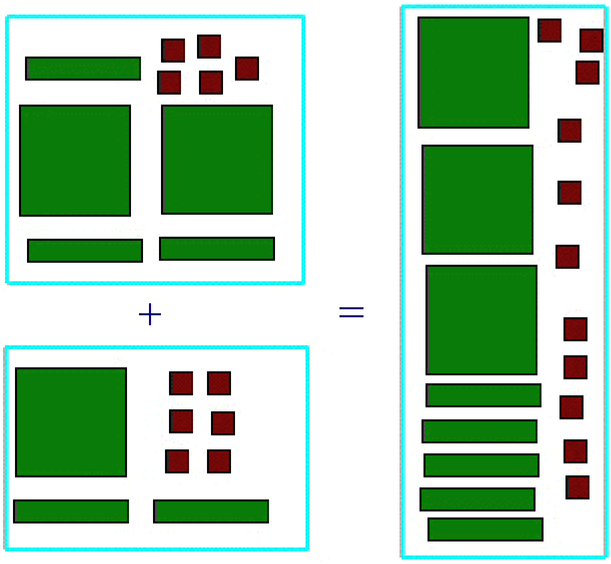
Collections, part 2

Combine the collection you made with a collection another group member made. How would you name the resulting collection? How would you draw or write the action of combining them?

Now make a collection and take some pieces out of it. How would you name the resulting collection? How would you draw or write the action of taking out pieces from a collection?

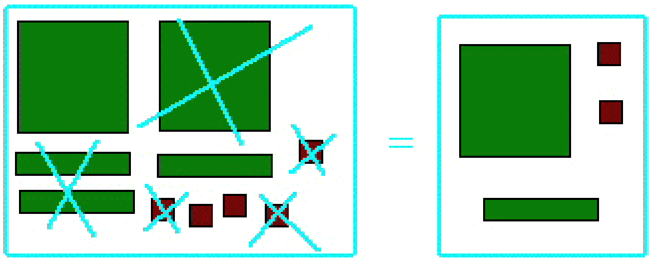
This is introducing the idea of adding and subtracting polynomials. Not much attention needs to be paid to this right now. All we are doing is agreeing on some more notational conventions:

$(2R^2 + 3R + 5) + (R^2 + 2R + 6) = (3R^2 + 5R + 11)$ *means*:



and

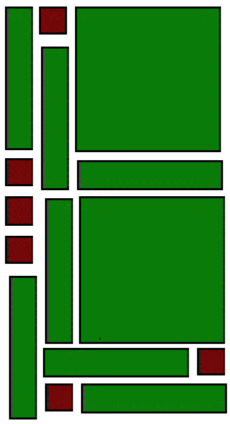
$(2R^2 + 3R + 5) - (R^2 + 2R + 3) = (R^2 + R + 2)$ *means*:

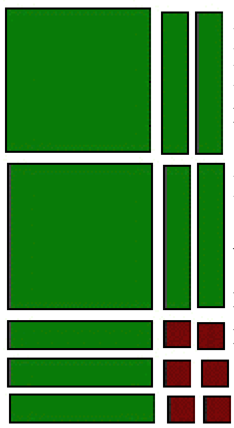


where crossing out the picture of a tile denotes removing it from the collection.

Again, have students perform the action on the tiles, and then draw the actions, and then write the symbols for the actions. If some of your lazier students want to stop drawing everything and just write symbols, that might be OK, but don't let them get the idea that drawing pictures and moving tiles is the "bad" way to do things.

Making rectangles.





Make a collection with two big squares, seven rectangles, and six nameless ones. How would you name that collection? Now arrange the collection so it makes a rectangle.

Both of these are rectangles made with the same pieces. I like the one on the right better. Why not introduce some aesthetic rules of Algebra tile rectangles? My first rule would be:

- Big squares can't touch little squares.

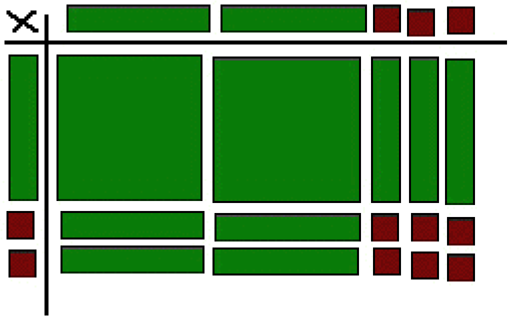
My second rule (which I am more lenient about) is:

- Little squares must all be together.

Measuring

Use rectangles and nameless ones to "measure" the length and width of your rectangles. For example:

the length and width of this rectangle are 2R+3 and R+2. It is sometimes confusing to have the measuring tiles so close to the rectangle. To clarify we can use "crossed lines" to separate them. (The X in the corner is because they are "crossed")



Spend some time on making rectangles, drawing them, measuring them, and writing down the collections and their measurements. For example, I could write $(2R^2+7R+6) = (R+2)x(2R+3)$ for the rectangle I made above.

If your students are getting tired of drawing you might introduce them to this notation:

X	2R	3
R	$2R^2$	3R
2	4R	6

This shows how the rectangle is made, and its measurements.

Now you have the idea of rectangles sorted it you might give the students a break and do something easy for a while:

Given two measurements, what tiles are in the rectangle? Start by putting the tiles for the measurements along your crossed lines, and then fill in the rectangle. Draw your result (or use a symbol table).

A pause to get our bearings.

You now know how to teach adding and subtracting polynomials (as long as there are no negatives involved), and factoring and multiplying polynomials with positive coefficients. Dividing is just factoring with one factor already known, so I skipped it. From here you can go a couple of directions.

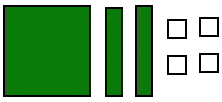
You can introduce a new variable, and pull out those other tiles. If you are using the cutouts you need to get the kids to choose a new label for one set to distinguish it from the other one. We're going to use them differently now, so it's time for a new name. You also need those squarish rectangles. What do they stand for?

Another direction you can go introduces negative coefficients. This is a bit tricky, so I'll head that way now, and let you figure out extra variables by yourself.

Back to square ones.

How would you take seven ones from three ones? Remember your work with integers. It's OK to turn the ones the other way up now. If you are using the commercial tiles they change colour, so you can use them for negative ones. If you are using the cutouts nothing happens at all, so this might be a good time to shade *one side* of each of your cut out tiles. Now, how do you find

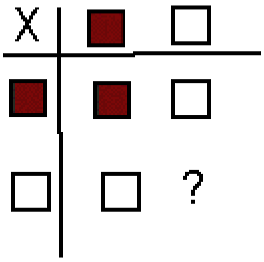
$(2R^2 - 3R - 7) - (R^2 - 5R - 3)$?

Sorry about all those symbols, but by now you and your students can probably handle them. Did everyone get ? Good.

Crossed-lines with a flip

Set up your crossed-lines and figure out how these new negative pieces work there when you multiply. Draw some examples, using symbols if you want. This part is pretty easy after you've mastered integers and multiplying polynomials with positive coefficients.

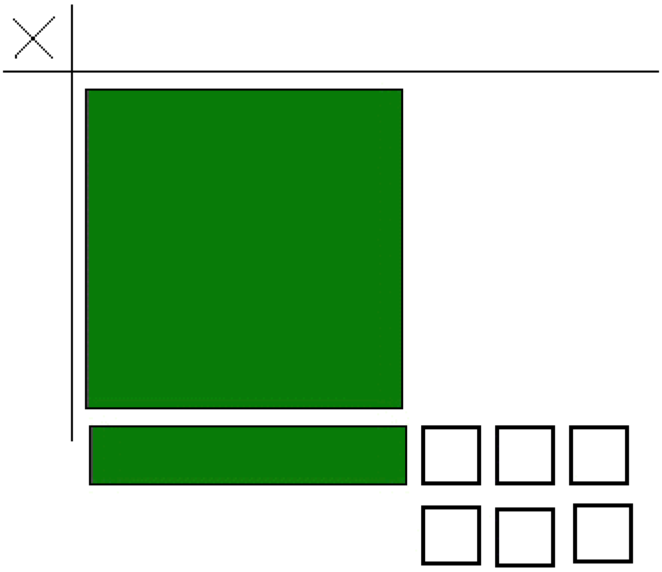
If your students are a bit shaky on multiplying integers you might want to remind them that *positive* numbers are used to count, so 1x1 means "one times one", which means one 1, which is 1; and 1x(-1) means one (-1) which is (-1). Once you have that sorted out, you can look at this multiplication of zero times zero (with zero repesented as (+1 + -1)):



Figuring out what to put in three of the four positions is easy. The fourth one is harder. But if we think about what we are multiplying (zero times zero) then we know what the answer is (zero), so the fourth position must be occupied by (+1).

Now for the tricky part

Let's factor $(R^2 + R - 6)$. Put the pieces between your crossed lines. There aren't enough. Try anyway, and put things where you learned they belonged when you were working with positive

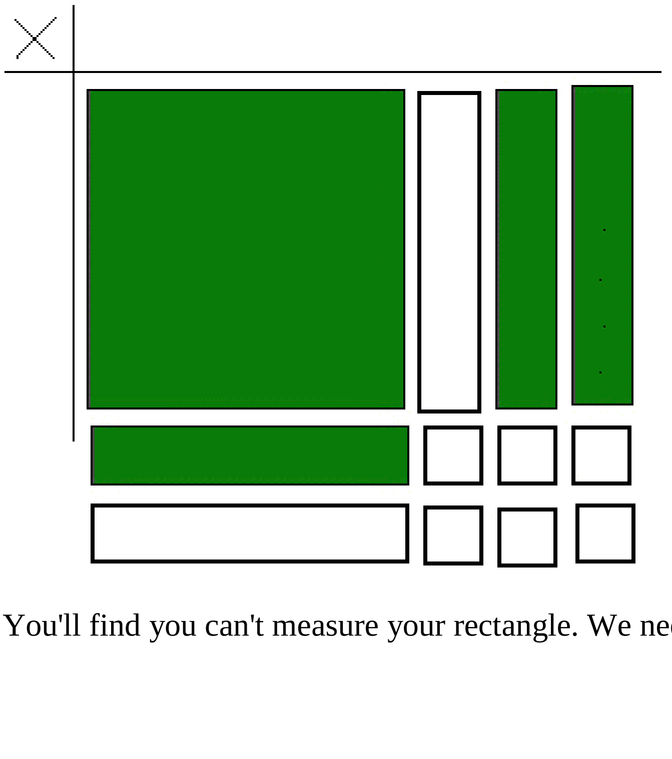


coefficients:

We know the R^2 and the nameless ones can't be in the same row or column because that violates one of our aesthetic rules. That leaves four spaces which look like they ought to have rectangles in them. Remember about adding zero? Try it. Now try some others.

Have students make up some by multiplying and give them to other people in their group.

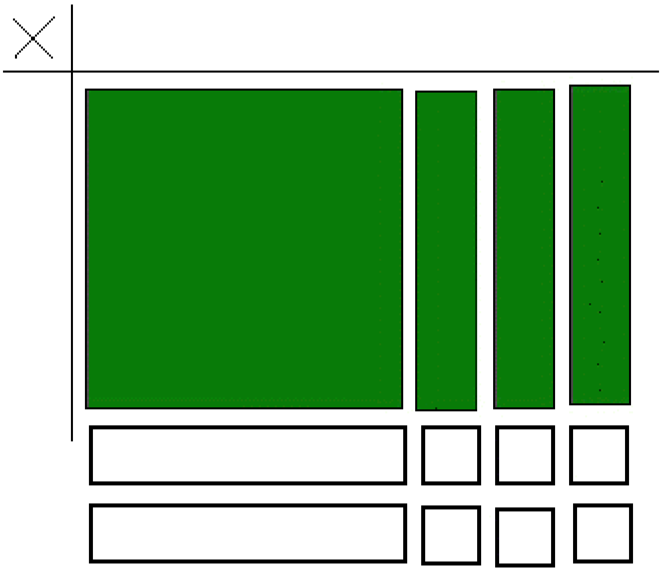
You might have discovered already that it isn't always clear where to put the rectangles you add. It's easy in the above example to know you need to add two positive rectangles and two negative rectangles, but you can't put them just anywhere. For example, if you put them like this:



You'll find you can't measure your rectangle. We need a new aesthetic rule:

- The squares in a column or row must either all be the same colour as the rectangles, or all be the other colour.

Given this rule, we can see that we should put our four new rectangles in like this:



In conclusion

Polynomials are unlike the other "numbers" students learn how to add, subtract, multiply and divide. They are not "counting" numbers. Giving polynomials a concrete referent (tiles) makes them real.

Implicit in the treatment I have given above is an important educational principle:

- Learning, at any level, is easiest if it starts with concrete actions, then moves to iconic representations of those actions, then to symbolic representations of those actions, and finally to formal actions on symbols.

In the case of the tiles these four stages are:

1. Actions on the tiles themselves
2. Drawings of those actions
3. Using symbols in or instead of the drawings
4. Traditional algebraic manipulations of symbols.