ΕΛΛΗΝΙΚΗ ΔΗΜΟΚΡΑΤΙΑ
Εθνικόν και Καποδιστριακόν
Πανεπιστήμιον Αθηνών

Ιστορία νεότερων Μαθηματικών

Ενότητα 3: Η Άλγεβρα της Αναγέννησης

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Περιγραφή Ενότητας

Ιταλοί Αβακιστές. Αλγεβρικός Συμβολισμός. Άλγεβρα στην Γαλλία, Γερμανία, Αγγλία. Εξισώσεις τρίτου και τετάρτου βαθμού. Μιγαδικοί αριθμοί. Εξισώσεις τετάρτου βαθμού και συμμετρίες.

Περιεχόμενα Υποενότητας

- Αλγεβρικός συμβολισμός και τεχνικές
- Euler και η συνεισφορά του
- Εξισώσεις μεγάλου βαθμού
- Απαλοιφή 2ου όρου

Η Άλγεβρα της Αναγέννησης

Αλγεβρικός συμβολισμός

Algebraic Symbolism And Technique (1/5)

- Άθροισμα εκθετών

Recall that Islamic algebra was entirely rhetorical. There were no symbols for the unknown or its powers nor for the operations performed on these quantities. Everything was written out in words. The same was generally true in the works of the early abacists and in the earlier Italian work of Leonardo of Pisa. Early in the fifteenth century, however, some of the abacists began to substitute abbreviations for unknowns. For example, in place of the standard words *cosa* (thing), *censo* (square), *cubo* (cube), and *radice* (root), some authors used the abbreviations *c*, *ce*, *cu*, and *R*. Combinations of these abbreviations were used for higher powers. Thus, *ce di ce* or *ce ce* stood for *censo di censo* or fourth power (x^2x^2); *ce cu* or *cu ce*, designating *censo*

Algebraic Symbolism And Technique (2/5)

- Γινόμενο εκθετών

di cubo and *cubo di censo*, respectively, stood for fifth power (x^2x^3); and *cu cu*, designating *cubo di cubo*, stood for sixth power (x^3x^3). By the end of the fifteenth century, however, the naming scheme for higher powers had changed, and authors used *ce cu* or *censo di cubo* to designate the sixth power ($(x^3)^2$) and *cu cu* or *cubo di cubo* to represent the ninth power ($(x^3)^3$). The fifth power was then designated as *p.r.* or *primo relato* and the seventh power as *s.r.* or *secondo relato*.

Pacioli, plus and minus

Near the end of the fifteenth century, Luca Pacioli introduced the abbreviations $\overline{p}$ and $\overline{m}$ to represent plus and minus (*più* and *meno*). (These particular abbreviations probably came from a more general practice of using the bar over a letter to indicate that some letters were missing.) As with other innovations, however, there was no great movement on the part of all the writers to use the same names or the same abbreviations. This change was a slow one. New symbols gradually came into use in the fifteenth and sixteenth centuries, but modern algebraic symbolism was not fully formed until the mid-seventeenth century.

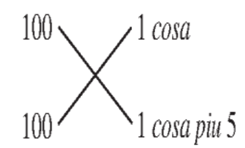
Algebraic Symbolism And Technique (3/5)

- $\left(\frac{100}{x}\right) + \frac{100}{x+5} = \frac{100(x+5) + 100x}{x(x+5)} = \frac{100x + 100*5 + 100x}{x^2 + 5x}$

Even without much symbolism, the Italian abacists, like their Islamic predecessors, were competent in handling operations on algebraic expressions. For example, Paolo Gerardi, in his *Libro di ragioni* of 1328, gave the rule for adding the fractions $100/x$ and $100/(x+5)$:

Algebraic Symbolism And Technique (4/5)

You place 100 opposite one *cosa* [x], and then you place 100 opposite one *cosa* and 5. Multiply crosswise as you see indicated, and you say . . . 100 times the one *cosa* that is across from it makes 100 *cose*. And then you say 100 times one *cosa* and 5 makes 100 *cose* and 500 in number. Now you must add one with the other which makes 200 *cose* and 500 in number. Then multiply one *cosa* times 1 *cosa* and 5 in number, making 1 *censo* [x^2] and 5 *cose*. Now you must divide 200 *cose* and 500 in number by one *censo* and 5 *cose* [$(200x + 500/(x^2 + 5x)$].[4]



Algebraic Symbolism And Technique (5/5)

- Rule of Signs Similarly, the rules of signs were also written out in words and even justified, here in a late fourteenth-century manuscript by an unknown author:

> Multiplying minus times minus makes plus. If you would prove it, do it thus: You must know that multiplying 3 and 3/4 by itself will be the same as multiplying 4 minus 1/4 [by itself]. Multiplying 3 and 3/4 by 3 and 3/4 gives 14 and 1/16. To multiply 4 minus 1/4 by 4 minus 1/4 . . . , say 4 by 4 is 16; now multiply across and say 4 times minus one quarter makes minus 4 quarters, which is minus 1, and 4 times minus 1/4 makes minus 1, so you have minus 2. Subtract this from 16 and it leaves 14. Now take minus 1/4 times minus 1/4. That gives 1/16, so one has the same as the other [multiplication].[5]

Euler, elements of algebra, c. 1765 (1/2)

- ***Elements of Algebra*** is an elementary mathematics textbook written by mathematician Leonhard Euler and originally published in 1765 in German.
- *Elements of Algebra* is one of the earliest books to set out algebra in the modern form we would recognize today (another early book being *Elements of Algebra* by Nicholas Saunderson, published in 1740), and is one of Euler's few writings, along with *Letters to a German Princess*, that are accessible to the general public. Written in numbered paragraphs as was common practice till the 19th century, *Elements* begins with the definition of mathematics and builds on the fundamental operations of arithmetic and number systems, and gradually moves towards more abstract topics.

Euler, elements of algebra, c. 1765 (2/2)

- In 1771, Joseph-Louis Lagrange published an addendum titled *Additions to Euler's Elements of Algebra*, which featured a number of important mathematical results. The original German title of the book was *Vollständige Anleitung zur Algebra*, which literally translates to *Complete Instruction to Algebra*.
- Two English translations are now extant, one by John Hewlett (1822), and the other, which is translated to English from a French translation of the book, by Charles Tayler (1824).
- Βλέπε και http://web.mat.bham.ac.uk/C.J.Sangwin/euler/index.html

Mathground, “Πληροφόριση”?

- http://mathground.net/leonard-eulers-elements-of-algebra-textbook/
- « The *Elements of Algebra* contains many important early results in mathematical analysis; for example, it contains Euler’s original proof of Fermat’s Last Theorem for the special case of $n = 3$.»

Euler's Elements of Algebra, multiplication part 1, art. 32

will produce.

32. Let us begin by multiplying $-a$ by 3 or $+3$. Now, since $-a$ may be considered as a debt, it is evident that if we take that debt three times, it must thus become three times greater, and consequently the required product is $-3a$. So if we multiply $-a$ by $+b$, we shall obtain $-ba$, or, which is the same thing, $-ab$. Hence we conclude, that if a positive quantity be multiplied by a negative quantity, the product will be negative; and it may be laid down

Euler's Elements of Algebra, multiplication
part 1, art. 33, minus times minus

as a rule, that + by + makes + or *plus*; and that, on the contrary, + by −, or − by +, gives −, or *minus*.

33. It remains to resolve the case in which − is multiplied by −; or, for example, $-a$ by $-b$. It is evident, at first sight, with regard to the letters, that the product will be ab; but it is doubtful whether the sign +, or the sign −, is to be placed before it; all we know is, that it must be one or the other of these signs. Now, I say that it cannot be the sign −: for $-a$ by $+b$ gives $-ab$, and $-a$ by $-b$ cannot produce the same result as $-a$ by $+b$; but must produce a contrary result, that is to say, $+ab$; consequently, we have the following rule: − multiplied by − produces +, that is, the same as + multiplied by + *.

Euler's Elements of Algebra, Surd Quantities or incommensurable, part 1, art. 128

pressed by fractions.

128. There is therefore a sort of numbers, which cannot be assigned by fractions, but which are nevertheless determinate quantities; as, for instance, the square root of 12: and we call this new species of numbers, *irrational numbers*. They occur whenever we endeavour to find the square root of a number which is not a square; thus, 2 not being a perfect square, the square root of 2, or the number which, multiplied by itself, would produce 2, is an irrational quantity. These numbers are also called *surd quantities*, or *incommensurables*.

Euler's Elements of Algebra, Surd Quantities or incommensurable, part 1, art. 132

produces a.

132. But when it is required to multiply $\sqrt{a}$ by $\sqrt{b}$, the product is $\sqrt{ab}$; for we have already shewn, that if a square has two or more factors, its root must be composed of the roots of those factors; we therefore find the square root of the product ab, which is $\sqrt{ab}$, by multiplying the square root of a, or $\sqrt{a}$, by the square root of b, or $\sqrt{b}$; &c. It is evident from this, that if b were equal to a, we should have $\sqrt{aa}$ for the product of $\sqrt{a}$ by $\sqrt{b}$. But $\sqrt{aa}$ is evidently a, since aa is the square of a.

Essays on the Theory of Numbers, by Richard Dedekind – Σύγκρουση γιγάντων? (1/2)

Just as addition is defined, so can the other operations of the so-called elementary arithmetic be defined, viz., the formation of differences, products, quotients, powers, roots, logarithms, and in this way we arrive at real proofs of theorems (as, e. g., $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$), which to the best of my knowledge have never been established before. The excessive length that is to be feared in the definitions of the more complicated operations is partly inherent in the nature of the subject but can for the most part be avoided. Very useful in this connection is the notion of an *interval*, i. e., a system A of rational numbers possessing the following characteristic property: if a and a' are numbers of the system A, then are all rational numbers lying between a and a' contained in A. The system R of all rational numbers, and also the two classes of any cut are intervals. If there exist a rational number a_1 which is less and a rational number a_2 which

Essays on the Theory of Numbers, by Richard Dedekind – Σύγκρουση γιγάντων? (2/2)

R of all rational numbers, and also the two classes of any cut are intervals. If there exist a rational number a_1 which is less and a rational number a_2 which is greater than every number of the interval A, then A is called a finite interval; there then exist infinitely many numbers in the same condition as a_1 and infinitely many in the same condition as a_2; the whole domain R breaks up into three parts A_1, A, A_2 and there enter two perfectly definite rational or irrational numbers α_1, α_2 which may be called respectively the lower and upper (or the less and greater) *limits* of the interval; the lower limit α_1 is determined by the cut for which the system A_1 forms the first class and the upper α_2 by the cut for which the system A_2 forms the second class. Of every rational or irrational number α lying between α_1 and α_2 it may be said that it lies *within* the interval A. If all numbers of an interval A are also numbers of an interval B, then A is called a portion of B.

Περαιτέρω βιβλιογραφία (1/2)

- Rule of radicals plhrofories genika, the origin of the problems in Euler’s algebra http://logica.ugent.be/albrecht/thesis/EulerProblems.pdf
- The Elementary Mathematical Works of Leonhard Euler (1707 – 1783) Paul Yiu Department of Mathematics: http://math.fau.edu/yiu/eulernotes99.pdf
- Euler’s “Mistake”? The Radical Product Rule in Historical Perspective https://webspace.utexas.edu/aam829/1/m/Euler_files/EulerMonthly.pdf
- HistoricaL conflicts and subtleties with the sign in textbooks, http://www.uv.es/gomezb/48historicalconflicts.pdf

Περαιτέρω βιβλιογραφία (2/2)

- Euler’s Elements of Algebra Gaps in Logic
- http://newmathdoneright.com/2012/08/01/eulers-elements-of-algebra-gaps-in-logic/
- Conflicts with the radical sign. A case study with patricia: http://www.uv.es/gomezb/36Conflicts.pdf
- A tale of two curricula: Euler's algebra text book: http://plus.maths.org/content/tale-two-curricula-eulers-algebra-text-book

Of Impossible, or Imaginary Quantities, which arise from the same source (1/3)

quantities, because they exist merely in the imagination.

144. All such expressions, as $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-4}$, &c. are consequently impossible, or imaginary numbers, since they represent roots of negative quantities; and of such numbers we may truly assert that they are neither nothing, nor greater than nothing, nor less than nothing; which necessarily constitutes them imaginary, or impossible.

Of Impossible, or Imaginary Quantities, which arise from the same source (2/3)

which necessarily constitutes them imaginary, or impossible.

145. But notwithstanding this, these numbers present themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them; since we know that by $\sqrt{-4}$ is meant a number which, multiplied by itself, produces -4; for this reason also, nothing prevents us from making use of these imaginary numbers, and employing them in calculation.

Of Impossible, or Imaginary Quantities, which arise from the same source (3/3)

and $\surd-16$ is equal to $4\surd-1$.

148. Moreover, as $\surd a$ multiplied by $\surd b$ makes $\surd ab$, we shall have $\surd 6$ for the value of $\surd-2$ multiplied by $\surd-3$; and $\surd 4$, or 2, for the value of the product of $\surd-1$ by $\surd-4$. Thus we see that two imaginary numbers, multiplied together, produce a real, or possible one.

Κείμενο του Euler με ριζικά

- Article 132, 133, 147
- Euler πιστεύει ότι $(\alpha\beta)^{1/2} = \alpha^{1/2}\beta^{1/2}$ και ότι π.χ. $(6)^{1/2} = (-2)^{1/2}(-3)^{1/2}$
- Όμως $(-2)^{1/2}(-3)^{1/2} = ((-1)2)^{1/2}((-1)3)^{1/2} =$
- $(-1)^{1/2}(2)^{1/2}\,(-1)^{1/2}(3)^{1/2}$
- $((-1)^{1/2})^2 2^{1/2} 3^{1/2} = (-1)(6)^{1/2}$
- $1^{1/2} = ((-1)(-1))^{1/2} = (-1)^{1/2}(-1)^{1/2} = ?$

Φύση του ορισμού

- Καταρχήν και καταρχάς αυθαίρετος
- Αν απαιτούμε ποιότητα, πρέπει να αναλογισθούμε την ιστορία και την νυν μαθηματική πράξη

Αιτιολόγηση Μιγαδικών αριθμών από Euler: πείθει? (1/2)

151. It remains for us to remove any doubt, which may be entertained concerning the utility of the numbers of which we have been speaking; for those numbers being impossible, it would not be surprising if they were thought entirely useless, and the object only of an unfounded speculation. This, however, would be a mistake; for the calculation of imaginary quantities is of the greatest importance, as questions frequently arise, of which we cannot immediately say whether they include any thing real and possible, or not; but when the solution of such a question leads to imaginary numbers, we are certain that what is required is impossible.

Αιτιολόγηση Μιγαδικών αριθμών από Euler: πείθει? (2/2)

impossible.

In order to illustrate what we have said by an example, suppose it were proposed to divide the number 12 into two such parts, that the product of those parts may be 40. If we resolve this question by the ordinary rules, we find for the parts sought $6+\sqrt{-4}$ and $6-\sqrt{-4}$; but these numbers being imaginary, we conclude, that it is impossible to resolve the question.

The difference will be easily perceived, if we suppose the question had been to divide 12 into two parts which multiplied together would produce 35; for it is evident that those parts must be 7 and 5.

Επαναλήψεις : Υπάρχουν οι Μιγαδικοί?

- Euler, ***Elements of Algebra, part 1, art. 151***

Algebraic Symbolism And Technique, Associativity

In general, the abacus manuscripts have lists of products and quotients of monomials written out, using the abbreviations for the powers of the unknown given above. But one fifteenth-century manuscript makes explicit the rules of exponents after having named the first nine powers of the unknown:

> If you wish to multiply these names [of the powers], . . . multiply the quantities [the coefficients of the powers] one into the other; then add together the degrees of the names and see the degree which is named. . . . If you wish to divide one of those names by another, it is necessary that what you wish to divide has a degree greater than that by which you wish to divide it. Make so: divide the quantities one by another. Afterwards, subtract the quantity of the degrees of those names one from another, . . . and [if] so many degrees remain, that quantity will be of so many degrees.[6]

Παραδείγματα

- $a^x α^{-y} = α^{x-y}$
- Προσεταιριστική ιδιότητα. Είναι προφανές?
- $a^3 = (aa)a = a(aa)$
- $α^4 = α^2 α^2$
- $(αα)α)α = (αα)(αα)$
- Σχόλιο. αν πρόσεχαν τέτοιες «λεπτομέριες»,τι?

Algebraic Symbolism And Technique: Αλλαγή μεταβλητών (1/2)

Antonio de' Mazzinghi (1353–1383), one of the few abacists about whom any biographical details are known, taught in the *Bottega d'abbaco* at the monastery of S. Trinita in Florence. His algebraic problems survive in several fifteenth-century manuscripts. Antonio was expert in devising clever algebraic techniques for solving complex problems. In particular, he explicitly used two different names for the two unknown quantities in many of these problems. For example, consider the following: "Find two numbers such that multiplying one by the other makes 8 and the sum of their squares is 27."[7] The abacist began the solution by supposing that the first number is *un cosa meno la radice d'alchuna quantità* (a thing minus the root of some quantity), while the second number equals *una cosa più la radice d'alchuna quantità* (a thing plus the root of some quantity). The two words *cosa* and *quantità* then serve in his rhetorical explication of the problem as the equivalent of our symbols x and y, that is, the first number is equal to $x - \sqrt{y}$, the second to $x + \sqrt{y}$.

Algebraic Symbolism And Technique: Αλλαγή μεταβλητών (2/2)

- $AB = 8,\ A^2 + B^2 = 27$
- $A = x + y^{1/2},\ B = x - y^{1/2},$
- $AB = x^2 - y = 8,\ A^2 + B^2 = 2(x^2 + y) = 27$

Higher Degree Equation (1/5)

The third major innovation of the Italian abacists was the extension of Islamic quadratic equation–solving techniques to higher-degree equations. In general, all of the abacists began their treatments of algebra by presenting al-Khwārizmī's six types of linear and quadratic equations and showing how each can be solved. But Maestro Dardi of Pisa in a work of 1344 extended this list to 198 types of equations of degree up to four, some of which involved

Higher Degree Equation (2/5)

radicals.[8] Most of the equations can be solved by a simple reduction to one of the standard forms, although in each case Dardi gave the solution anew, presenting both a numerical example and a recipe for solving the particular type of equation. For example, he noted that the equation $ax^4 = bx^3 + cx^2$ has the solution given by

$$x = \sqrt{\left(\frac{b}{2a}\right)^2 + \frac{c}{a}} + \frac{b}{2a},$$

that is, it has the same solution as the standard equation $ax^2 = bx + c$. (Note that 0 is never considered as a solution.) Similarly, the equation $n = ax^3 + \sqrt{bx^3}$ can be solved for x^3 by reducing it to a quadratic equation in $\sqrt{x^3}$.

Higher Degree Equation (3/5)

More interesting than these quadratic equations are four examples of irreducible cubic and quartic equations. Dardi's cubic equation was $x^3 + 60x^2 + 1200x = 4000$. His rule tells us to divide 1200 by 60 (giving 20), cube the result (which gives 8000), add 4000 (giving 12,000), take the cube root ($\sqrt[3]{12,000}$), and finally subtract the quotient of 1200 by 60. Dardi's answer, which is correct, was that $x = \sqrt[3]{12,000} - 20$. If we write this equation using modern notation and then give Dardi's solution rule, we obtain the solution to the equation $x^3 + bx^2 + cx = d$ in the form

$$x = \sqrt[3]{\left(\frac{c}{b}\right)^3 + d} - \frac{c}{b}.$$

Higher Degree Equation (4/5)

It is easy enough to see that this solution is wrong in general, and Dardi even admitted as much. How then did Dardi figure out the correct solution to his particular case? We can answer this question by considering the problem that illustrates the rule, a problem in compound interest: A man lent 100 *lire* to another and after 3 years received back a total of 150 *lire* in principal and interest, where the interest was compounded annually. What was the interest rate? Dardi set the rate for 1 *lira* for 1 month at x *denarii*. Then the annual interest on 1 *lira* is $12x$ *denarii* or $(1/20)x$ *lire*. So the amount owed after 1 year is $100(1 + x/20)$ and after 3 years is $100(1 + x/20)^3$. Dardi's equation therefore is

$$100\left(1+\frac{x}{20}\right)^3 = 150 \quad \text{or} \quad 100 + 15x + \frac{3}{4}x^2 + \frac{1}{80}x^3 = 150$$

or, finally,

$$x^3 + 60x^2 + 1200x = 4000.$$

Higher Degree Equation (5/5)

Because the left side of this equation comes from a cube, it can be completed to a cube once again by adding an appropriate constant. In general, because $(x+r)^3 = x^3 + 3rx^2 + 3r^2x + r^3$, to complete $x^3 + bx^2 + cx$ to a cube, we must find r satisfying two separate conditions, $3r = b$ and $3r^2 = c$, conditions that can only be satisfied when $b^2 = 3c$. In Dardi's example, with $b = 60$ and $c = 1200$, the condition is satisfied and $r = c/b = 20$.

Dardi gave a similar rule for solving special quartic equations, while Piero della Francesca (c. 1420–1492), more famous as a painter than as an abacist, extended these rules to fifth- and sixth-degree equations in his own *Trattato d'abaco*. Neither man stated explicitly that the rules apply only to the cases reducible to the form $h(1+x)^n = k$, where $n = 4, \ 5, \ 6$. There is

Απαλοιφή δεύτερου όρου (1/3)

$n = 4, 5, 6$. There is another (anonymous) manuscript of this period which suggests that the equation $x^3 + px^2 = q$ can be solved by setting $x = y - \frac{p}{3}$ where y is a solution of $y^3 = 3\left(\frac{p}{3}\right)^2 y + \left[q - 2\left(\frac{p}{3}\right)^3\right]$. This is correct as far as it goes, but the author has only managed to replace one cubic equation by another. In the numerical example presented,

Απαλοιφή δεύτερου όρου (2/3)

he solves the new equation by trial, but this could also have been done with the original. Nevertheless, although the abacists did not manage to give a complete general solution to the cubic equation, they, like their Islamic predecessors, wrestled with the problem and arrived at partial results, as noted in the opening quotation from the work of Luca Pacioli (1445–1517).

Απαλοιφή δεύτερου όρου (3/3)

- $a_n x^n + \alpha_{n-1} x^{n-1} + \alpha_{n-2} x^{n-2} + \cdots + a_0 = 0$
- $x = y - (\alpha_{n-1})/(nan)$
- $a_n x^n + \alpha_{n-1} x^{n-1} + +\alpha_{n-2} x^{n-2} + \cdots + a_0 = an(y - (\alpha_{n-1})/(n\,an))^n +$
- $\alpha_{n-1}(y - (\alpha_{n-1})/(n\,an))^{n-1} + o\rho o\iota\ \beta\alpha\theta\mu o\upsilon < (\nu - 2) =$
- $= an(y^n - n((\alpha_{n-1})/(an)n)yn^{-1} + o\rho o\iota\ \beta\alpha\theta\mu o\upsilon < (\nu - 2)) +$
- $\alpha_{n-1} y^{n-1} + o\rho o\iota\ \beta\alpha\theta\mu o\upsilon < (\nu - 2) =$
- $a_n y^n + = a_n y^n + o\rho o\iota\ \beta\alpha\theta\mu o\upsilon < (\nu - 2)$

Τέλος Υποενότητας

Αλγεβρικός συμβολισμός

Χρηματοδότηση

- Το παρόν εκπαιδευτικό υλικό έχει αναπτυχθεί στο πλαίσιο του εκπαιδευτικού έργου του διδάσκοντα.
- Το έργο «**Ανοικτά Ακαδημαϊκά Μαθήματα στο Πανεπιστήμιο Αθηνών**» έχει χρηματοδοτήσει μόνο την αναδιαμόρφωση του εκπαιδευτικού υλικού.
- Το έργο υλοποιείται στο πλαίσιο του Επιχειρησιακού Προγράμματος «Εκπαίδευση και Δια Βίου Μάθηση» και συγχρηματοδοτείται από την Ευρωπαϊκή Ένωση (Ευρωπαϊκό Κοινωνικό Ταμείο) και από εθνικούς πόρους.

Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης

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Σημείωμα Ιστορικού Εκδόσεων Έργου

Το παρόν έργο αποτελεί την έκδοση 1.0.

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Διατήρηση Σημειωμάτων

Οποιαδήποτε αναπαραγωγή ή διασκευή του υλικού θα πρέπει να συμπεριλαμβάνει:

- το Σημείωμα Αναφοράς
- το Σημείωμα Αδειοδότησης
- τη δήλωση Διατήρησης Σημειωμάτων
- το Σημείωμα Χρήσης Έργων Τρίτων (εφόσον υπάρχει)

μαζί με τους συνοδευόμενους υπερσυνδέσμους.

Σημείωμα Χρήσης Έργων Τρίτων

Το Έργο αυτό κάνει χρήση των ακόλουθων έργων:

Εικόνες/Σχήματα/Διαγράμματα/Φωτογραφίες