Μάθημα 8ο
(23/03/2015)

Κανόνας της Αλυσιδωτής Παραγώγισης / Κανόνας Αλυσίδα

$f : (\alpha, \beta) \to (\gamma, \delta)$

$g : (\gamma, \delta) \to \mathbb{R}$

$$\begin{tikzcd} x \arrow[r, "f"] \arrow[dr, "g \circ f"'] & f(x) \arrow[d, "g"] \\ & {} \end{tikzcd}$$

$\exists\, f'(x_0)\,,\ g'(f(x_0))$

$(g \circ f)'(x_0) = g'(f(x_0)) \cdot f'(x_0)$

$J_{x_0}(g \circ f) = J_{f(x_0)}(g) \cdot J_{x_0}(f)$ (Γραμμική Άλγεβρα)

$d(g \circ f)(x_0) = dg(f(x_0)) \circ df(x_0)$

$(d\varphi(x_0)(h) = \varphi'(x_0) \cdot h\,,\ h \in \mathbb{R})$

Θεώρημα. Διαφόρισης της σύνθεσης 2 συναρτήσεων.

$\vec{f} : A\ (\subseteq \mathbb{R}^d) \longrightarrow B\ (\subseteq \mathbb{R}^m)$

$\vec{g} : B \longrightarrow \mathbb{R}^\ell$ όπου A, B ανοιχτά

Υποθέτουμε ότι για το $\vec{x_0} \in A$ $\exists\, d\vec{f}(\vec{x_0})$ και $d\vec{g}(\vec{f}(\vec{x_0}))$.

Τότε $\exists\, d(\vec{g} \circ \vec{f})(\vec{x_0}) = d\vec{g}(\vec{f}(\vec{x_0})) \circ d\vec{f}(\vec{x_0})$.

Απόδειξη

$\exists\, d\vec{f}(\vec{x_0}) : \vec{f}(\vec{x_0}+\vec{h}) = \vec{f}(\vec{x_0}) + \vec{T_1}(\vec{h}) + \|\vec{h}\| \cdot \vec{q_1}(\vec{h})$

όπου $\vec{T_1} = d\vec{f}(\vec{x_0})$, $\lim\limits_{\vec{h}\to\vec{0}} \vec{q_1}(\vec{h}) = \vec{q_1}(\vec{0}) = \vec{0}$

$(\|\vec{h}\| < r_1,\ S(\vec{x_0}, r_1) \subseteq A)$

$\exists\, d\vec{g}(\vec{f}(\vec{x_0})) : \vec{g}(\vec{f}(\vec{x_0}) + \vec{t}) = \vec{g}(\vec{f}(\vec{x_0})) + \vec{T_2}(\vec{h}) + \|\vec{t}\| \cdot \vec{q_2}(\vec{t})$

όπου $\vec{T_2} = d\vec{g}(\vec{f}(\vec{x_0}))$, $\lim\limits_{\vec{t}\to\vec{0}} \vec{q_2}(\vec{t}) = \vec{q_2}(\vec{0}) = \vec{0}$

$(\|\vec{t}\| < r_2,\ S(\vec{f}(\vec{x_0}), r_2) \subseteq B)$

. $\vec{h_n} \in S(\vec{0}, r_1)$, $\vec{h_n} \neq \vec{0}$, $\vec{h_n} \xrightarrow[n]{} \vec{0}$

Ορίζουμε $\vec{t_n} = \vec{T_1}(\vec{h_n}) + \|\vec{h_n}\| \vec{q_1}(\vec{h_n}) \xrightarrow[n]{} \vec{0}$, $t_n \in S(\vec{0}, r_2)$ $n \geq n_0$.

Παρατηρούμε ότι $\|\vec{t_n}\| \leq \|\vec{T_1}(\vec{h_n})\| + \|\vec{q_1}(\vec{h_n})\| \cdot \|\vec{h_n}\| \leq M \|\vec{h_n}\| + \|\vec{q_1}(\vec{h_n})\| \cdot \|\vec{h_n}\|$

$\left(\dfrac{\|\vec{t_n}\|}{\|\vec{h_n}\|} \right)_{n\in\mathbb{N}}$ είναι φραγμένη.

$\vec{g} \circ \vec{f}(\vec{x_0} + \vec{h}) = \vec{g}(\vec{f}(\vec{x_0} + \vec{h_n})) = \vec{g}\circ\vec{f}(\vec{x_0}) + \vec{T_2}(\vec{t_n}) + \|\vec{t_n}\| \vec{q_2}(\vec{t_n})$

$(\vec{T_2}$ γραμμική$) = \vec{g}\circ\vec{f}(\vec{x_0}) + \vec{T_2}\circ\vec{T_1}(\vec{h_n}) + \|\vec{h_n}\| \vec{T_2}(\vec{q_1}(\vec{h_n})) + \|t_n\| \, \|\vec{q_2}(t_n)\|$

Άρα $\dfrac{\vec{g}\circ\vec{f}(\vec{x_0}+\vec{h}) - \vec{g}\circ\vec{f}(\vec{x_0}) - \vec{T_2}\circ\vec{T_1}(\vec{h})}{\|\vec{h_n}\|} = \vec{T_2}(\vec{q_1}(\vec{h_n})) + \dfrac{\|\vec{t_n}\|}{\|\vec{h_n}\|} \cdot \|\vec{q_2}(t_n)\| \xrightarrow[n]{} \vec{0}$

Επομένως, ΑΜ. $\exists \lim\limits_{\vec{h}\to\vec{0}} \dfrac{\vec{g}\circ\vec{f}(\vec{x_0}+\vec{h}) - \vec{g}\circ\vec{f}(\vec{x_0}) - d\vec{g}(\vec{f}(\vec{x_0})) \circ d\vec{f}(\vec{x_0})}{\|\vec{h}\|} = 0$

Από τη μοναδικότητα της συν. του διαφορικού,

$\Longrightarrow \exists\, d(\vec{g}\circ\vec{f})(\vec{x_0}) = d\vec{g}(\vec{f}(\vec{x_0})) \circ d\vec{f}(\vec{x_0})$

Πόρισμα

$\vec f : A(\subseteq \mathbb{R}^d) \longrightarrow B(\subseteq \mathbb{R}^m)$, $\vec f = (f_1, \dots, f_m)$

$\vec g : B \longrightarrow \mathbb{R}^\ell$, $\vec g = (g_1, \dots, g_\ell)$

$$\begin{tikzcd} \mathbb{R}^d \arrow[r, "\vec f"] \arrow[dr, "\vec g \circ \vec f"'] & \mathbb{R}^m \arrow[d, "\vec g"] \\ & \mathbb{R}^\ell \end{tikzcd}$$

$\vec h = \vec g \circ \vec f = (h_1, \dots, h_\ell)$ / $\exists\, d\vec f(\bar x_0)$, $\exists\, d\vec g(\vec f(\bar x_0))$

Τότε $J_{\bar x_0}(\vec g \circ \vec f) = J_{\vec f(\bar x_0)}(\vec g) \cdot J_{\bar x_0}(\vec f)$

Ιδιαιτέρως Εάν $d = 1 = \ell$ Τότε

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\vec f = \vec r"] \arrow[dr, "g \circ \vec r"'] & \mathbb{R}^m \arrow[d, "g"] \\ & \mathbb{R} \end{tikzcd}$$

$$\boxed{\frac{d(g\circ\vec r)(t_0)}{dt} = \nabla \vec g(\vec r(t_0)) \cdot \vec r\,'(t_0)}$$

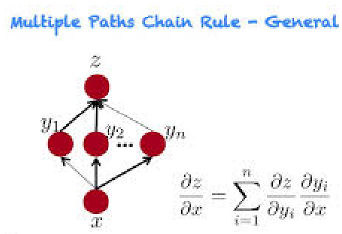
Απόδειξη Θεώρημα + Γραμμική Άλγεβρα

Αναλυτικά:

$$\begin{pmatrix} \frac{\partial h_1}{\partial t_1} & \frac{\partial h_1}{\partial t_2} & \cdots & \frac{\partial h_1}{\partial t_d} \\ \vdots & & & \\ \frac{\partial h_\ell}{\partial t_1} & \frac{\partial h_\ell}{\partial t_2} & \cdots & \frac{\partial h_\ell}{\partial t_d} \end{pmatrix} (\bar x_0)$$

$$= \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_m} \\ \vdots & & & \\ \frac{\partial g_\ell}{\partial x_1} & \frac{\partial g_\ell}{\partial x_2} & \cdots & \frac{\partial g_\ell}{\partial x_m} \end{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial t_1} & \frac{\partial f_1}{\partial t_2} & \cdots & \frac{\partial f_1}{\partial t_d} \\ \vdots & & & \\ \frac{\partial f_m}{\partial t_1} & & \cdots & \frac{\partial f_m}{\partial t_d} \end{pmatrix}$$

$$\frac{\partial h_i(\bar x_0)}{\partial t_j} = \sum_{k=1}^{m} \frac{\partial g_i(\vec f(\bar x_0))}{\partial x_k} \cdot \frac{\partial f_k(\bar x_0)}{\partial t_j} \qquad \begin{array}{l} i = 1, 2, \dots, \ell \\ j = 1, \dots, d \end{array}$$



Ασκήσεις

1. $\vec{f}(u,v) = (u+v\,,\,u\cdot v)$ $\qquad f_1(u,v) = u+v,\quad f_2(u,v) = u\cdot v$

$\vec{g}(x,y) = (x^2+y^2, x^3, x+y)$ $\qquad g_1(x,y) = x^2+y^2,\quad g_2(x,y) = x^3,$

$g_3(x,y) = x+y$

$\vec{h} = \vec{g}\circ\vec{f}$. $\vec{h} = (h_1, h_2, h_3)$

$$\begin{tikzcd} \mathbb{R}^2 \ni (u,v) \arrow[r, "\vec{f}"] \arrow[dr, "\vec{h}"'] & \mathbb{R}^2\ (x,y) \arrow[d, "\vec{g}"] \\ & \mathbb{R}^3 \end{tikzcd}$$

$\dfrac{\partial h_1}{\partial u}$, $\dfrac{\partial h_3}{\partial v}$ Κατ' ευθεία ΚΑΙ

με τον Κανόνα της Αλυσίδας

Λύση

$\vec{h}(u,v) = \vec{g}(u+v, u\cdot v) = \left((u+v)^2 + (u\cdot v)^2,\ (u+v)^3,\ u+v+u\cdot v\right)$

$h_1(u,v) = (u+v)^2 + (u\cdot v)^2$

$h_2(u,v) = (u+v)^3$

$h_3(u,v) = u+v+u\cdot v$

$$\begin{pmatrix} \frac{\partial h_1}{\partial u} & \frac{\partial h_1}{\partial v} \\ \frac{\partial h_2}{\partial u} & \frac{\partial h_2}{\partial v} \\ \frac{\partial h_3}{\partial u} & \frac{\partial h_3}{\partial v} \end{pmatrix}_{(u,v)} = \begin{pmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \\ \frac{\partial g_3}{\partial x} & \frac{\partial g_3}{\partial y} \end{pmatrix}_{(\vec{f}(u,v))} \cdot \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix}$$

- $\dfrac{\partial h_1}{\partial u} = 2(u+v) + 2uv^2 \quad (1)$

- $\dfrac{\partial h_2}{\partial u} = \dfrac{\partial g_1}{\partial x}\cdot\dfrac{\partial f_1}{\partial u} + \dfrac{\partial g_1}{\partial y}\cdot\dfrac{\partial f_2}{\partial u} = 2(u+v)\cdot 1 + 2uv\cdot v \quad (2)$

$\underline{(1) = (2)}$

Όμοια $\dfrac{\partial h_3}{\partial v} = \dfrac{\partial g_3}{\partial x}\cdot\dfrac{\partial f_1}{\partial v} + \dfrac{\partial g_3}{\partial y}\cdot\dfrac{\partial f_2}{\partial v}$

2) $f: \mathbb{R}^d \setminus \{0\} \longrightarrow \mathbb{R}$ διαφορίσιμη

Ομογενής α-τάξης $(\alpha \neq 0)$ $f(\lambda \vec{x}) = \lambda^{\alpha} \cdot f(\vec{x})$, $\lambda > 0$
$\vec{x} \neq \vec{0}$.

ΝΔΟ $\vec{x} \cdot \nabla f(\vec{x}) = a \cdot f(\vec{x})$, $\vec{x} \in \mathbb{R}^d \setminus \{\vec{0}\}$.

$\left(\sum_{i=1}^{d} x_i \frac{\partial f(\vec{x})}{\partial x_i} = a f(\vec{x}) \right)$ / Εξίσωση Euler για ομογενείς συναρτήσεις

Πολύ σημαντική!

Παλιό Θέμα.

Λύση

$\vec{x} \in \mathbb{R}^d \setminus \{\vec{0}\}$ σταθερό.

$\vec{r}(\lambda) = \lambda \cdot \vec{x}$, $\lambda > 0$.

$\vec{r}'(\lambda) = \vec{x}$, $\lambda > 0$

$\longrightarrow h = f \circ \vec{r}$ διαφορίσιμη.

$$\frac{dh(t)}{dt} = \nabla f(\vec{r}(t)) \circ \vec{r}'(t) \quad (1)$$

$h(t) = f \circ \vec{r}(t) = f(t\vec{x}) = t^{\alpha} \cdot f(\vec{x})$, $t > 0$

$$\frac{dh(t)}{dt} = \alpha t^{\alpha - 1} f(\vec{x}), \ t > 0 \quad (2)$$

Από (1) και (2) $\nabla f(\vec{r}(t)) \cdot \vec{x} = a t^{\alpha-1} \cdot f'(\vec{x})$, $t > 0$.

$t = 1 > 0$ $\vec{x} \cdot \nabla f(\vec{x}) = a \cdot f(\vec{x})$

Σημείωση: Ισχύει και το αντίστροφο,
δηλαδή, αν f = διαφορίσιμη $\mathbb{R}^d \setminus \{\vec{0}\}$ και $\vec{x} \circ \nabla f(\vec{x}) = a \cdot f(\vec{x})$
$\Rightarrow f$ = ομογενής α-τάξης

Υπόδειξη. $\varphi(t) = t^{-\alpha} f(t\vec{x})$

3) i) $Q(\vec{x}) = \vec{x}\,\Pi\,\vec{x}^{\perp}$ θετική

Τετραγωνική μορφή, $\vec{x}\,\Pi\,\vec{x}^{\perp} > 0$, $\vec{x} \neq \vec{0}$

ΝΔΟ $f(\vec{x}) = [\vec{Q}(\vec{x})]^{p/2}$ $(p \neq 0)$

ικανοποιεί τη εξίσωση $\vec{x} \cdot \nabla f(\vec{x}) = p f(\vec{x})$, $\vec{x} \neq \vec{0}$

ii) $f : \mathbb{R}^2 \setminus \{(0,0)\} \longrightarrow \mathbb{R}$, C^2 a-ομογενείς

ΝΔΟ οι $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ είναι ομογενείς στον $\mathbb{R}^2 \setminus \{(0,0)\}$

και $x^2 \cdot \frac{\partial^2 f(x,y)}{\partial x^2} + 2xy \frac{\partial^2 f(x,y)}{\partial x \partial y} + y^2 \frac{\partial^2 f(x,y)}{\partial y^2} = a(\alpha-1) f(x)$

Λύση. i) $f(\lambda \vec{x}) = [(\lambda\vec{x})\Pi(\lambda\vec{x})^{\perp}]^{p/2} = \lambda^p f(\vec{x})$ $\vec{x} \neq \vec{0}$, $\lambda > 0$.

Από Άσκηση 1, ισχύει $\vec{x} \cdot \nabla f(\vec{x}) = p\, f(\vec{x})$, $\vec{x} = \vec{0}$.

ii) $\lambda > 0$, $\vec{x} \neq \vec{0}$, $\frac{\partial f(\lambda x, \lambda y)}{\partial x} = \lim_{\lambda h \to 0} \frac{f(\lambda x + \lambda h, \lambda y) - f(\lambda x, \lambda y)}{\lambda h} =$

$= \frac{\lambda^{\alpha}}{\lambda} \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lambda^{\alpha-1} \frac{\partial f}{\partial x}(x,y)$ $\lambda > 0$.

Το ίδιο ισχύει και για $\frac{\partial f}{\partial y}$.

$\frac{\partial f}{\partial x}$ ομογενής τάξης $(\alpha-1)$ $\xRightarrow{\text{Ασκ. 2}} (x,y) \cdot \nabla\left(\frac{\partial f}{\partial x}(x,y)\right) = (\alpha-1)\frac{\partial f}{\partial x}(x,y)$

$x \frac{\partial^2 f}{\partial x^2} + y \frac{\partial^2 f}{\partial y \partial x} = (\alpha-1) \frac{\partial f}{\partial x}$ επί x (1)

$\frac{\partial f}{\partial y}$ ομογενής τάξης $(\alpha-1)$ $\xRightarrow{\text{Ασκ. 2}} (x,y) \cdot \nabla\left(\frac{\partial f}{\partial y}(x,y)\right) = (\alpha-1)\frac{\partial f}{\partial y}(x,y)$

$x \frac{\partial^2 f}{\partial y \partial x} + y \frac{\partial^2 f}{\partial y^2} = (\alpha-1) \frac{\partial f}{\partial y}$ επί y. (2)

① + ② : $x^2 \dfrac{\partial^2 f}{\partial x^2} + xy \dfrac{\partial^2 f}{\partial y \partial x} + xy \dfrac{\partial^2 f}{\partial x \partial y} + y^2 \dfrac{\partial^2 f}{\partial y^2} = (\alpha - 1)\left[x \dfrac{\partial f}{\partial x} + y \dfrac{\partial f}{\partial y}\right]$

(Θ. Clairaut) $x^2 \dfrac{\partial^2 f}{\partial x^2} + 2xy \dfrac{\partial^2 f}{\partial x \partial y} + y^2 \dfrac{\partial^2 f}{\partial y^2} = \alpha(\alpha - 1) f(x,y).$

$\downarrow$

$\vec{x} \cdot \nabla f(\vec{x}) = \alpha f(\vec{x})$

4) $g : \mathbb{R} \setminus \{0\} \longrightarrow \mathbb{R}$ $g - C^2$

και $V(x,y,z) = g(\sqrt{x^2+y^2+z^2})$ $(\vec{r} = (x,y,z),\ r = \|\vec{r}\|,\ V(\vec{r}) = g(r))$.

i) Τελεστής Laplace του V.

ii) Εάν V = αρμονική, $\lim\limits_{r \to +\infty} g(r) = 0$, να ευρεθεί το V

Λύση <u>Παλιό Θέμα</u>.

$$\begin{tikzcd} (x,y,z) \arrow[r, "f"] \arrow[dr, "V"'] & \sqrt{x^2+y^2+z^2} = t \arrow[d, "g"] \\ & \mathbb{R} \end{tikzcd}$$

$$\frac{\partial V}{\partial x} = \frac{\partial f}{\partial x} \cdot \frac{dg(t)}{dt}$$

$$= \frac{x}{\sqrt{x^2+y^2+z^2}} \cdot g'(\sqrt{x^2+y^2+z^2})$$

$$\frac{\partial^2 V(x,y,z)}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}} \cdot g'(\sqrt{x^2+y^2+z^2})\right)$$

$$\frac{\partial}{\partial x}\left(\frac{x}{(x^2+y^2+z^2)^{1/2}}\right) = \frac{(x^2+y^2+z^2)^{1/2} - \dfrac{x^2}{(x^2+y^2+z^2)^{1/2}}}{(x^2+y^2+z^2)} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$$

$$\frac{\partial g'(\sqrt{x^2+y^2+z^2})}{\partial x} = \frac{x}{\sqrt{x^2+y^2+z^2}}\, g''(t)$$

$$\begin{tikzcd} (x,y,z) \arrow[r] \arrow[dr] & \sqrt{x^2+y^2+z^2} = t \arrow[d, "g'"] \\ & \mathbb{R} \end{tikzcd}$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}} \cdot g'(t) + \frac{x}{(x^2+y^2+z^2)^{3/2}} \cdot \frac{x}{(x^2+y^2+z^2)^{1/2}} \cdot g''(t)$$

$$\frac{\partial^2 V}{\partial x^2} = \frac{y^2+z^2}{r^3} \cdot g'(r) + \frac{x^2}{r^2} \cdot g''(r).$$

$$\Delta V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{y^2+z^2}{r^3} \cdot g'(r) + \frac{x^2}{r^2} g''(r)$$

$$+ \frac{x^2+z^2}{r^3} g'(r) + \frac{y^2}{r^2} \cdot g''(r) + \frac{x^2+y^2}{r^3} g'(r) + \frac{z^2}{r^2} g''(r)$$

$$= g''(r) + \frac{2r^2}{r^3} g'(r) = g''(r) + \frac{2}{r} g'(r), \quad r \neq 0.$$

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ii) $\Delta V = 0$

$$g''(r) + \frac{2}{r} g'(r) = 0 \Longrightarrow r^2 g''(r) + 2r g'(r) = 0 \Longrightarrow$$

$$\frac{d}{dr}\left(r^2 \cdot g'(r)\right) = 0 \Longrightarrow r^2 \cdot g'(r) = c_1 \Longrightarrow g'(r) = \frac{c_1}{r^2}$$

$$\Longrightarrow g(r) = -\frac{c_1}{r} + c' \Big/ \lim_{r \to +\infty} g(r) = 0$$

$$\Longrightarrow g(r) = -\frac{c_1}{r}$$

Άρα $V(x,y,z) = -\dfrac{c_1}{\sqrt{x^2+y^2+z^2}}$

Θεώρημα Μέσης Τιμής Δ. Λ.

$f: A (\subseteq \mathbb{R}^d) \longrightarrow \mathbb{R}$ A = ανοιχτό, f διαφορ. στο A.

$\vec{\alpha}, \vec{\beta} \in A$, $\vec{\alpha} \neq \vec{\beta}$: $[\vec{\alpha}, \vec{\beta}] \subseteq A$

Τότε $\exists\, \vec{z} \in [\vec{\alpha}, \vec{\beta}] \setminus \{\vec{\alpha}, \vec{\beta}\} : f(\vec{\beta}) - f(\vec{\alpha}) = \nabla f(\vec{z}) \cdot (\vec{\beta} - \vec{\alpha})$

Απόδειξη

$\vec{r}(t) = \vec{\alpha} + t \cdot (\vec{\beta} - \vec{\alpha})$, $t \in [0,1]$

$[\vec{\alpha}, \vec{\beta}] = \vec{r}([0,1])$

$\vec{r}(1) = \vec{\beta}$, $\vec{r}(0) = \vec{\alpha}$.

$h = f \circ \vec{r} : [0,1] \longrightarrow \mathbb{R}$, διαφορίσιμη.

$h(1) - h(0) = h'(t_0) \cdot (1-0) = h'(t_0)$

ΘΜΤ για κάποιο $t_0 \in (0,1)$

$f \circ \vec{r}(1) - f \circ \vec{r}(0) = \dfrac{d(f \circ \vec{r})(t_0)}{dt}$

$f(\vec{\beta}) - f(\alpha) = \nabla f(\vec{r}(t_0)) \cdot (\vec{\beta} - \vec{\alpha})$

$\vec{z} = \vec{r}(t_0)$.

Σημειώσεις

1. Δεν ισχύει ΘΜΤ για διανυσματικές συναρτήσεις

πχ. $\vec{r}(t) = (\sigma\upsilon\nu t, \eta\mu t)$, $t \in [0, 2\pi]$

$\vec{r}(2\pi) = \vec{r}(0) = (1,0)$

$\vec{r}\,'(t) = (-\eta\mu t, \sigma\upsilon\nu t)$

$\vec{r}(2\pi) - \vec{r}(0) = (0,0) \neq r'(t) \cdot 2\pi \quad \forall t \in [0, 2\pi]$.

2. Δεν υπάρχουν Κανόνες L' Hospital για συναρτήσεις 2 (ή περισσοτέρων) μεταβλητών. —

$$\left(\frac{f(\vec{x})-f(\vec{\alpha})}{g(\vec{x})-g(\vec{\alpha})} = \frac{\nabla f(\vec{\xi})\cdot(\vec{x}-\vec{\alpha})}{\nabla g(\vec{\xi})\cdot(\vec{x}-\alpha)} \right.$$

Δεν γίνεται απλοποίηση !!)

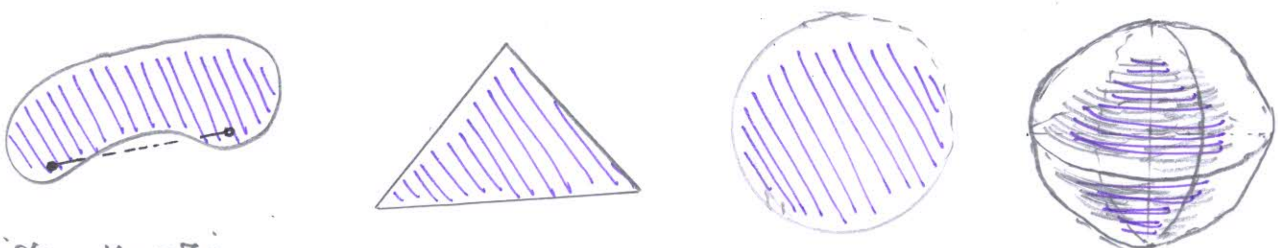
→ ΔΕΝ κάνουμε L'Hospital

Ορισμοί

$K \subseteq \mathbb{R}^d$, $K \neq \emptyset$

i) K κυρτό $\Longleftrightarrow$ $\vec{\alpha}, \vec{\beta} \in K$ έχουμε ότι $[\vec{\alpha}, \vec{\beta}] \subseteq K$

$\Longleftrightarrow$ $\vec{\alpha}, \vec{\beta} \in K$ $\vec{\alpha} + t(\vec{\beta} - \vec{\alpha}) \subseteq K$, $t \in [0,1]$



όχι κυρτό

κυρτά σύνολα

ii) K πολυγωνικά συνεκτικό $\Longleftrightarrow$ $\vec{a}, \vec{\beta} \in K$

$\Pi = [\vec{\alpha} = \vec{a}_1, \vec{a}_2] \cup [\vec{a}_2, \vec{\alpha}_3] \cup \dots \cup [\vec{\alpha}_K, \vec{\alpha}_{K+1} = \vec{\beta}]$

$\Pi \subseteq K$

πολυγωνική γραμμή

iii) K κατά τόξα συνεκτικό $\Longleftrightarrow$ $\vec{\alpha}, \vec{\beta} \in K$, $\vec{\alpha} \neq \vec{\beta}$

$\exists\ \vec{r} : [\alpha, \beta] \to K$ συνεχής

$\vec{r}(\alpha) = \vec{\alpha}$, $\vec{r}(\vec{\beta}) = \vec{\beta}$

Ισχύει.

- $K \subseteq \mathbb{R}$

K κυρτό $\Longleftrightarrow$ K πολ. συνεκτικό $\Longleftrightarrow$ K κτ συνεκτικό

- $K \subseteq \mathbb{R}^d$, $d \geq 2$

K κυρτό $\underset{\not\Longleftarrow}{\Longrightarrow}$ K πολ. συνεκτικό $\underset{\not\Longleftarrow}{\Longrightarrow}$ K κτ συνεκτικό

Ισχύει.

K = ανοικτό + κτ. συνεκτικό $\Longrightarrow$ K = πολυγωνικά συνεκτικό.