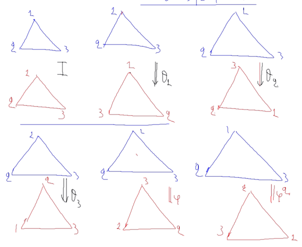
Βασική άλγεβρα
Δευτέρα 3/11/2014



Έχουμε την ομάδα $\{I, \theta_1, \theta_2, \theta_3, \varphi, \varphi^2\}$

ονομάζεται διεδρική ομάδα του ισοπλεύρου τριγ. D_3

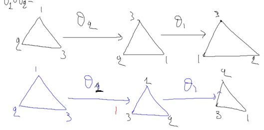
άρα $|D_3| = 6$

$I^{-1} = I,\ \theta_1^{-1} = \theta_1,\ \theta_2^{-1} = \theta_2,\ \theta_3^{-1} = \theta_3,\ \varphi^{-1} = \varphi^2$

$(\varphi^2)^{-1} = \varphi$

$\varphi^3 = I$

$\theta_1 \circ \theta_2 =$



Άρα $\theta_1 \circ \theta_2 = \varphi$

και $\theta_2 \circ \theta_1 = \varphi^2$

Τελικά $\theta_1 \circ \theta_2 \neq \theta_2 \circ \theta_1$ άρα η D_3 μη-αβελιανή

$S_3 = \{\theta : \{1,2,3\} \to \{1,2,3\},\ \theta \text{ 1-1 και επί}\}$

$I = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix},\ \delta_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},\ \delta_2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

$\delta_3 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix},\ \rho = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},\ \rho^2 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$

$S_3 = \{I, \delta_1, \delta_2, \delta_3, \rho, \rho^2\}$

$\{I, (23), (13), (12), (123), (132)\} = S_3$

$(123) = (231) = (312)$

$I^{-1} = I,\ (23)$

$(23)(23) = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = I$, $\quad (123)(123) = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} = (132) = \rho^2$

$(23)^2 = I$

$(12)^2 = I,\ (13)^2 = I$

$\underbrace{(123)}_{\rho}\underbrace{(132)}_{\rho^2} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = I$

$\rho\rho^2 = \rho^3 = I$

$D_3 = \{I, \theta_1, \theta_2, \theta_3, \varphi, \varphi^2\}$

$S_3 = \{I, \delta_1, \delta_2, \delta_3, \rho, \rho^2\}$ Ισομορφισμοί ομάδων

$(\mathbb{Z}_6, +) = (\langle 1 (\text{mod}\,6) \rangle, +)$ κυκλική τάξης 6

Θεώρημα Έστω Γ ομάδα τάξης 6. Τότε

είτε Γ αβελιανή και $\Gamma \cong (\mathbb{Z}_6, +)$

είτε Γ όχι αβελ. και $\Gamma \cong D_3 \cong S_3$

Αν $D_3 \cong \mathbb{Z}_6$

$\omega : D_3 \to \mathbb{Z}_6$

$\alpha, \beta \in D_3$ με $\alpha\beta \neq \beta\alpha$

$\Rightarrow \alpha\beta(\beta\alpha)^{-1} \neq I$

$\to \alpha\beta\alpha^{-1}\beta^{-1} \neq I$

$\omega(\alpha\beta\alpha^{-1}\beta^{-1}) =$

$\omega(\alpha)\cdot\omega(\beta)\cdot\omega(\alpha^{-1})\cdot\omega(\beta^{-1})$

$= I = \omega(I)$

Απόδ. $\Gamma = \{e, \alpha, \beta, \gamma, \delta, \varepsilon\}$

$\tau(\alpha) = 6 \Rightarrow \alpha \neq e, \alpha^2 \neq e, \alpha^3 \neq e, \alpha^4 \neq e, \alpha^5 \neq e, \alpha^6 = e$

$G = \{e, \alpha, \alpha^2, \alpha^3, \alpha^4, \alpha^5\}$

$\tau(\alpha) = 2,\ \alpha^2 = e, \beta^2 = e, \gamma^2 = e, \delta^2 = e, \varepsilon^2 = e$

$\tau(\alpha) = 3$

$\Rightarrow \alpha = \alpha^{-1}, \beta = \beta^{-1}, \gamma = \gamma^{-1}, \delta = \delta^{-1}, \varepsilon = \varepsilon^{-1}$

$\alpha\beta = \alpha^{-1}\beta^{-1} = (\beta\alpha)^{-1} = \beta\alpha$

δηλ. η Γ είναι αβελ.

τότε το $H = \{e, \alpha, \beta, \alpha\beta\}$ είναι υποομάδα τάξης 4 άτοπο. όταν Γ με τάξη 6

Αν όλα τα στοιχεία είναι τάξης 3

$\alpha \neq e, \alpha^2 \neq e, \alpha^3 = e$

$\langle \alpha \rangle = \{e, \alpha, \alpha^2\}$

Θεωρούμε $\beta \in \Gamma$ με $\beta \notin \{e, \alpha, \alpha^2\}$

αφού β τάξης 3 $\Longrightarrow \langle \beta \rangle = \{e, \beta, \beta^2\}$

$\langle \alpha \rangle \cap \langle \beta \rangle = \{e\}$

$e, \alpha, \alpha^2, \underline{\beta}, \beta\alpha, \underline{\beta\alpha^2}, \beta^2, \beta^2\alpha, \beta^2\alpha^2$ άτοπο.

$\alpha = \alpha^2 \Rightarrow \alpha = e$ άτοπο

$\beta\alpha = \beta^2\alpha^2 \Rightarrow \alpha = \beta$

Τελικά σε μια ομάδα Γ τάξης 6 υπάρχει
(τουλάχιστον ένα) στοιχείο $\boxed{\alpha \text{ τάξης } 2}$
και (τουλ. ένα) " " $\boxed{\beta \text{ τάξης } 3}$

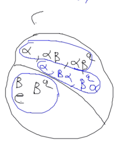
$\alpha\beta \in \Gamma$

- $\alpha\beta = \beta\alpha$, $(\alpha\beta)^1 = \alpha\beta$
  $\alpha\beta = e \Rightarrow \beta = \alpha^{-1} = \alpha$
  $(\alpha\beta)^2 = \alpha\beta\alpha\beta = \alpha^2\beta^2 = \beta^2 \neq e$
  $(\alpha\beta)^3 = \alpha^3\beta^3 = \alpha \neq e$ άρα $\text{τάξη}(\alpha\beta) = 6 \Longrightarrow \Gamma$ κυκλ. $\cong (\mathbb{Z}_6, +)$
- $\alpha\beta \neq \beta\alpha$

$\langle \beta \rangle = \{e, \beta, \beta^2\}$, $\beta^3 = e$

$\alpha\langle \beta \rangle = \{\alpha \cdot e, \alpha \cdot \beta, \alpha\beta^2\}$

$\langle \beta \rangle \cdot \alpha = \{e \cdot \alpha, \beta \cdot \alpha, \beta^2 \cdot \alpha\}$



$\langle \beta \rangle$

δηλ. $\{\alpha, \alpha\beta, \alpha\beta^2\} = \{\alpha, \beta\alpha, \beta^2\alpha\}$

άρα $\alpha\beta = \beta^2\alpha$ και
$\beta\alpha = \alpha\beta^2$

συνεπώς $\Gamma \cong D_3 \cong S_3$

<u>Άσκηση</u> Να μελετηθεί η ομάδα $(U(\mathbb{Z}_7), \cdot)$

<u>Λύση</u>

$U(\mathbb{Z}_7) = \{1 (\text{mod } 7), 2, 3, 4, 5, 6\}$

$2, 2^2, 2^3$

$3, 3^2, 3^3, 3^6$

$(\mathbb{Z}_2, +, \cdot)$

$A = \{M \in \mathbb{Z}_2^{2\times 2}\}$ δηλαδή $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

$(U(A), \cdot)$ $\qquad |A| = 2^4 = 16$ στοιχ.

$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ $\qquad \mathbb{Z}_2^{\nu\times\nu}$

$\{\cdot\}$ $\qquad \mathbb{Z}_2^2 \to \mathbb{Z}_2^2$ ν-άδα-2

$3 \cdot 2 = 6$ $\qquad \{(0,0), (0,1), (1,0), (1,1)\}$

$\left\{ \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$

$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$