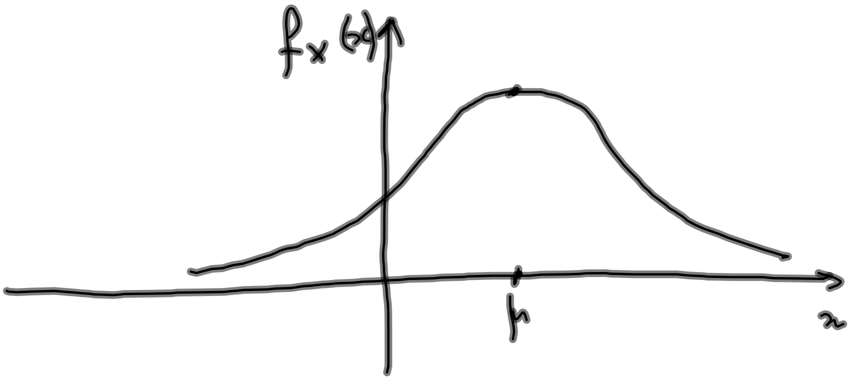
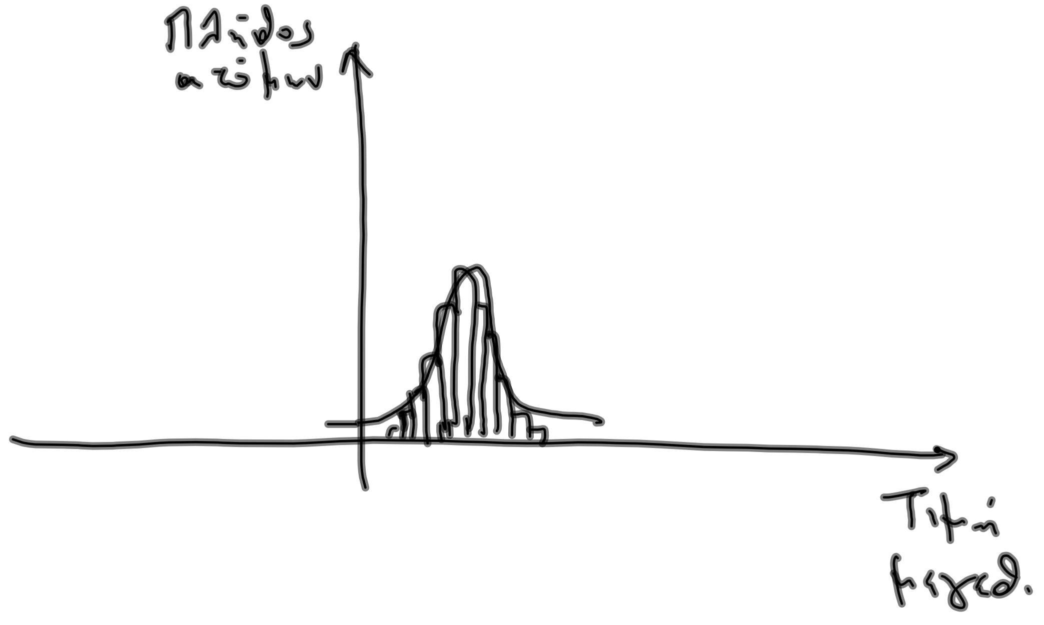
Η κανονική κατανομή

X τ.μ. συνεχής

σππ $f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $x \in \mathbb{R}$



Σε στατιστικές για διάφορα μεγέθη
π.χ. ύψος, βάρος, δείκτης νοημοσ. κλπ.



Κανονική κατανομή

συνδέεται με το Κεντρικό Οριακό Θεώρ

$$\left.\begin{array}{l} X_1, X_2, \ldots, X_n \text{ ανεξ.} \\ E[X_i]=\mu \\ Var[X_i]=\sigma^2 \end{array}\right\} \Rightarrow \sum_{i=1}^{n} X_i \sim N(n\mu, n\sigma^2)$$

μεγάλο n

$(n \geq 30)$

Τυποποίηση μιας κανονικής τ.μ

$X \sim \mathcal{N}(\mu, \sigma^2) \qquad (E[X] = \mu, \ Var[X] = \sigma^2)$

$$Z = \frac{X - \mu}{\sigma} = \frac{1}{\sigma} \cdot X - \frac{\mu}{\sigma}$$

$$E[Z] = \frac{1}{\sigma} E[X] - \frac{\mu}{\sigma} = \frac{\mu}{\sigma} - \frac{\mu}{\sigma} = 0$$

$$Var[Z] = \frac{1}{\sigma^2} Var[X] = \frac{1}{\sigma^2} \cdot \sigma^2 = 1$$

Άρα $Z \sim \mathcal{N}(0,1) \leftarrow$ Τυποποιημένη κανονική

Τυποποιημένη κανονική

σππ $N(\mu, \sigma^2)$

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \quad x \in \mathbb{R}$$

$$\Downarrow \; \mu = 0, \; \sigma = 1$$

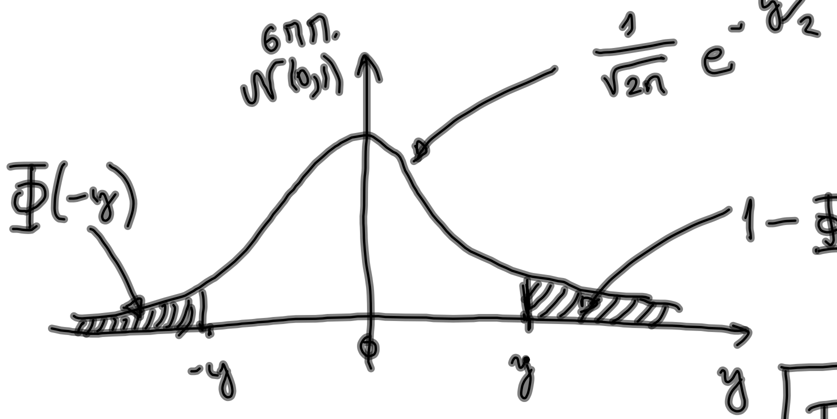
σππ $N(0,1)$ σκ $N(0,1)$

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \; x \in \mathbb{R} \Rightarrow F_X(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt, \; x \in \mathbb{R}$$

Πίνακες της $\Phi(y) \leftarrow$ σκ της $N(0,1)$

$\boxed{\text{Πίνακας περιέχει } \Phi(y),\ 0 \le y \le 3}$ (με βήμα 0.01)

$\Phi(3) \simeq 0.997$ οπότε $\boxed{\Phi(y) \simeq 1,\ y \ge 3}$



$$\boxed{\Phi(-y) = 1-\Phi(y)}$$

Παράδειγμα υπολογ. πιθ. $\mathcal{N}(\mu, \sigma^2)$

$X \sim \mathcal{N}(\mu, \sigma^2)$

$$P(a \le X \le b) = P(X \le b) - P(X \le a)$$
$$= F_X(b) - F_X(a) \leftarrow \text{Προβλ.}$$

$$P(\underbrace{a \le X \le b}_{\sim\, \mathcal{N}(\mu,\sigma^2)}) = P\left(\frac{a-\mu}{\sigma} \le \underbrace{\frac{X-\mu}{\sigma}}_{\sim\, \mathcal{N}(0,1)} \le \frac{b-\mu}{\sigma}\right)$$

$$= \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right).$$

Παρ: Ύψος μονόπατωσης

$$X = \text{Ύψος χιον} \sim N(60, 20^2)$$
$$\qquad \qquad \qquad \qquad \quad \underset{\mu}{\shortparallel} \qquad \underset{\sigma^2}{\shortparallel}$$

$$P(X \geq 80) = P\left(\frac{X-\mu}{\sigma} \geq \frac{80-\mu}{\sigma}\right)$$
$$= P\left(\underset{\underset{N(0,1)}{\sim}}{Z} \geq \frac{80-60}{20}\right)$$
$$= P(Z \geq 1)$$
$$= 1 - P(Z \leq 1) = 1 - \Phi(1)$$
$$= 1 - 0.8413 = 0.1587$$

Ασκήσεις στις συνεχείς τ.μ.

1) Υπολογισμοί πιθ., μέσων τιμών, διασπορών για συνεχή τ.μ. ***

σππ. $f_X(x) = \begin{cases} cx(3-x), & 0 \leq x \leq 3 \\ 0, & \text{διαφ.} \end{cases}$

$c \geq 0$ $\quad \int_{-\infty}^{\infty} f_X(x)\,dx = 1 \Longrightarrow$

$$1 = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^3 c x (3-x)\,dx$$

$$= c\int_0^3 (3x - x^2)\,dx = c\left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_{x=0}^{3}$$

$$= c\left[\frac{27}{2} - \frac{27}{3}\right] = c\cdot\frac{27}{6}$$

$$\Downarrow$$

$$c = \frac{6}{27}$$

$$f_X(x) = \begin{cases} \frac{6}{27}x(3-x), & 0 \leq x \leq 3 \\ 0 & , \text{διαφ.} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\,dx = \int_0^3 x \cdot \frac{6}{27} x(3-x)\,dx$$

$$= \int_0^3 \left(\frac{18}{27} \cdot x^2 - \frac{6}{27} \cdot x^3\right) dx$$

$$= \left[\frac{18}{27} \cdot \frac{x^3}{3} - \frac{6}{27} \cdot \frac{x^4}{4}\right]_{x=0}^{3}$$

$$= \frac{18}{\cancel{27}} \cdot \frac{\cancel{27}}{3} - \frac{6}{27} \cdot \frac{81}{4} = \cdots = 1.5.$$

Δεκ 3-4:22 μμ

$$\mathrm{Var}[X] = E[X^2] - (E[X])^2.$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$= \int_0^3 x^2 \cdot \frac{6}{27} x(3-x)\,dx$$

$$= \cdots$$

$$P(X \geq 1 \mid X \in [-1,2])$$

$$= \frac{P(X \geq 1,\ -1 \leq X \leq 2)}{P(-1 \leq X \leq 2)}$$

$$= \frac{P(1 \leq X \leq 2)}{P(-1 \leq X \leq 2)}$$

$$= \frac{\int_1^2 f_X(x)\,dx}{\int_{-1}^2 f_X(x)\,dx}$$

$$= \frac{\int_1^2 \frac{6}{27} x(3-x)\,dx}{\int_0^2 \frac{6}{27} x(3-x)\,dx}$$

$$= \dots$$

$$P(X \geq 1 \mid X \in [0,4])$$
$$= P(X \geq 1)$$
$$= \int_1^\infty f_X(x)\,dx$$
$$= \int_1^3 \frac{6}{27} x(3-x)\,dx$$
$$= \dots$$

$$P(X \geq 1 \mid 0 \leq X \leq 4)$$
$$= \frac{P(1 \leq X \leq 4)}{P(0 \leq X \leq 4)}$$
$$= \frac{\int_1^3 \frac{6}{27} x(3-x)\,dx}{1}$$

Δεκ 3-4:33 μμ

② Υπολογισμοί για συνάρτηση συνεχούς τ.μ.

X σ.π.π. $\quad f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{διαφορ.} \end{cases}$

$Y = X^2 \quad Z = e^X$

$$F_Y(y) = P(Y \le y) = P(X^2 \le y) = \begin{cases} 0, & y<0 \\ P(-\sqrt{y} \le X \le \sqrt{y}), & y \ge 0 \end{cases}$$

$y \in \mathbb{R}$

$$F_Y(y) = \begin{cases} 0, & y < 0 \\ P(-\sqrt{y} \leq X \leq \sqrt{y}), & y \geq 0 \end{cases}$$

$$= \begin{cases} 0, & y < 0 \\ \int_{\cancel{-\sqrt{y}}\,0}^{\sqrt{y}} f_X(x)\,dx, & y \geq 0 \end{cases}$$

$$= \int_{-\sqrt{y}}^{0} f_X(x)\,dx + \int_{0}^{\sqrt{y}} f_X(x)\,dx$$

$$= \begin{cases} 0, & y < 0 \\ \int_0^{\sqrt{y}} 1\,dx = \sqrt{y}, & 0 \leq y \leq 1 \\ \int_0^1 1\,dx = 1, & y > 1 \end{cases}$$

$$F_Y(y) = \begin{cases} 0, & y<0 \\ \sqrt{y}, & 0 \le y \le 1 \\ 1, & y>1 \end{cases}$$

$$f_Y(y) = \frac{dF_Y(y)}{dy} = \begin{cases} 0, & y<0 \\ \frac{1}{2\sqrt{y}}, & 0 \le y \le 1 \\ 0, & y>1 \end{cases}.$$

Δεκ 3-4:47 μμ

$$E[Y] = E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\,dx$$

$$= \int_0^1 x^2 \cdot 1\,dx = \left[\frac{x^3}{3}\right]_{x=0}^{1} = \frac{1}{3}.$$

$$E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\,dy = \int_0^1 y \cdot \frac{1}{2\sqrt{y}}\,dy$$

$$= \int_0^1 \frac{\sqrt{y}}{2}\,dy = \left[\frac{y^{\frac{3}{2}}}{\cancel{2} \cdot \frac{3}{\cancel{2}}}\right]_{y=0}^{1} = \frac{1}{3}.$$

Δεκ 3-4:54 μμ

Δεκ 3-4:57 μμ