Exercises

1. Describe a set A that satisfies properties (i) and (ii) of the induction principle (see Section 1.1.2, page 17) and is a *proper* superset of $\mathbb{N}$.

2. When we do an induction proof, is it necessary to prove the basis before we prove the Induction Step, or could we prove the two steps in either order?

3. Let $P(n)$ be a predicate of natural numbers. Suppose we prove the following facts:

- $P(0)$ holds
- $P(1)$ holds
- for any $i \geq 0$, if $P(i)$ holds then $P(i+2)$ holds

Does this constitute a valid proof that $P(n)$ holds for all $n \in \mathbb{N}$? Justify your answer.

4. Let $P(n)$ be a predicate of the integers. Suppose we prove the following facts:

- $P(0)$ holds
- for any $i \geq 0$, if $P(i)$ holds then $P(i+1)$ holds
- for any $i \geq 0$, if $P(i)$ holds then $P(i-1)$ holds

Does this constitute a valid proof that $P(n)$ holds for all $n \in \mathbb{Z}$? Justify your answer.

5. Let $P(n)$ be a predicate of the integers. Suppose we prove the following facts:

- $P(0)$ holds
- for any $i \geq 0$, if $P(i)$ holds then $P(i+1)$ holds
- for any $i \leq 17$, if $P(i)$ holds then $P(i-1)$ holds

Does this constitute a valid proof that $P(n)$ holds for all $n \in \mathbb{Z}$? Justify your answer.

6. Let $P(n)$ be a predicate of the integers. Suppose we prove the following facts:

- $P(17)$ holds
- for any $i \geq 17$, if $P(i)$ holds then $P(i+1)$ holds
- for any $i \leq -17$, if $P(i)$ holds then $P(i-1)$ holds

Does this constitute a valid proof that $P(n)$ holds, for all integers n such that $n \geq 17$ or $n \leq -17$? Justify your answer.

7. Use induction to prove that, for any integers $m \geq 2$ and $n \geq 1$, $\sum_{t=0}^{n} m^t = \frac{m^{n+1}-1}{m-1}$.

8. Review the definitions of equality between sequences and the subsequence relationship between sequences. Then prove that any two *finite* sequences are equal if and only if each is a subsequence of the other. Does this result hold for *infinite* sequences? (Show that it does, or provide a counterexample proving that it does not.)

9. Prove that every nonempty finite set of natural numbers has a maximum element. Does the same hold for infinite sets of natural numbers? Compare this with the Well-Ordering principle.

10. Use Proposition 1.6 to prove Propositions 1.3 and 1.5 without using induction.

11. Each of the proofs of Proposition 1.9 (see Example 1.7, page 30, and Example 1.12, page 40) not only proves that we can make postage for any amount of $n \geq 18$ cents using only 4-cent and 7-cent stamps; it also (implicitly) provides a recursive algorithm for determining *how* to do so. Specifically, starting from any $n \geq 18$ and "unwinding the induction" backwards (either in steps of 1 — as in the first proof — or in steps of 4 — as in the second proof) we can see how to recursively compute the number of 4-cent and the number of 7-cent stamps for making exactly n cents of postage. Do the two algorithms produce the same answer for each amount of postage? That is, for any value of $n \geq 18$, do both algorithms yield the same number of 4-cent and 5-cent stamps with which to make n cents of postage? Justify your answer!

Note that, in principle, this need not be the case. For instance, postage of 56 cents can be produced in any of the following ways:

- seven 4-cent stamps and four 7-cent stamps
- fourteen 4-cent stamps
- eight 7-cent stamps.

12. Use complete induction to prove that for each nonempty full binary tree the number of leaves exceeds the number of internal nodes by exactly one.

13. Let $P(n)$ be the predicate:

$P(n)$: postage of exactly n cents can be made using only 4-cent and 6-cent stamps.

Consider the following complete induction "proof" of the statement "$P(n)$ holds for all $n \geq 4$".

BASIS: $n = 4$. Postage of exactly 4 cents can be made using just a single 4-cent stamp. So $P(4)$ holds, as wanted.

INDUCTION STEP: Let $i \geq 4$ be an arbitrary integer, and suppose that $P(j)$ holds for all j such that $4 \leq j < i$. That is, for all j such that $4 \leq j < i$, postage of exactly j cents can be made using only 4-cent and 6-cent stamps. We must prove that $P(i)$ holds. That is, we must prove that postage of exactly i cents can be made using only 4-cent and 6-cent stamps.

Since $i - 4 < i$, by induction hypothesis we can make postage of exactly $i - 4$ cents using only 4-cent and 6-cent stamps. Suppose this requires k 4-cent stamps and ℓ 6-cent stamps; i.e., $i - 4 = 4 \cdot k + 6 \cdot \ell$. Let $k' = k + 1$ and $\ell' = \ell$. We have

$$\begin{aligned}
4 \cdot k' + 6 \cdot \ell' &= 4 \cdot (k + 1) + 6 \cdot \ell && \text{[by definition of } k' \text{ and } \ell' \text{]} \\
&= 4 \cdot k + 6 \cdot \ell + 4 \\
&= (i - 4) + 4 && \text{[by induction hypothesis]} \\
&= i
\end{aligned}$$

Thus, $P(i)$ holds, as wanted.

Clearly, however, we can't make an odd amount of postage using only 4-cent and 6-cent stamps! Thus, the statement "$P(n)$ holds for all $n \geq 4$" is certainly false. Consequently, the above "proof" is incorrect. What is wrong with it?

14. Define the sequence of integers $a_0, a_1, a_2, \cdots$ as follows:

$$a_i = \begin{cases} 2, & \text{if } 0 \leq i \leq 2 \\ a_{i-1} + a_{i-2} + a_{i-3}, & \text{if } i > 2 \end{cases}$$

Use complete induction to prove that $a_n < 2^n$, for every integer $n \geq 2$.

15. An n-bit *Gray code* is a sequence of all 2^n n-bit strings with the property that any two successive strings in the sequence, as well as the first and last strings, differ in exactly one position. (You can think of the 2^n strings as arranged around a circle, in which case we can simply say that any two successive strings on the circle differ in exactly one bit position.) For example, the following is a 3-bit Gray code: $111, 110, 010, 011, 001, 000, 100, 101$. There are many other 3-bit Gray codes — for example, any cyclical shift of the above sequence, or reversal thereof, is also a 3-bit Gray code.

Prove that for every integer $n \geq 1$, there is an n-bit Gray code.

16. Let n be any positive integer. Prove that

(a) Every set S that contains binary strings of length n such that no two strings in S differ in *exactly* one position, contains no more than 2^{n-1} strings.

(b) There exists a set S that contains exactly 2^{n-1} binary strings of length n such that no two strings in S differ in exactly one position.