

Voronoi diagram and Delaunay triangulation

Ioannis Emiris & Vissarion Fisikopoulos

Dept. of Informatics & Telecommunications, University of Athens



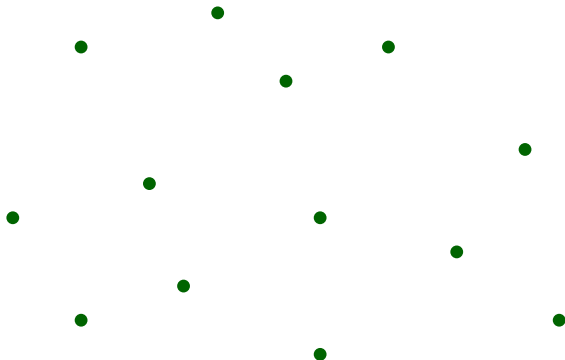
Computational Geometry, spring 2015

Outline

Outline

Example and definition

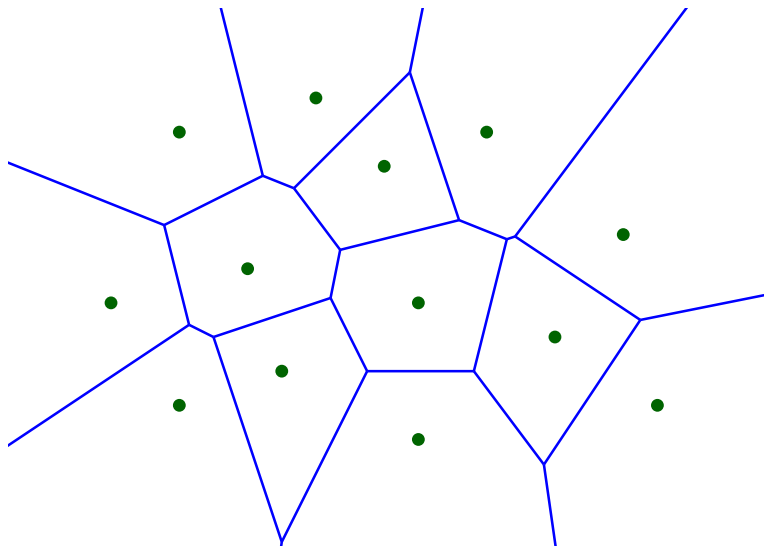
Sites: $P := \{p_1, \dots, p_n\} \subset \mathbb{R}^2$



Example and definition

Sites: $P := \{p_1, \dots, p_n\} \subset \mathbb{R}^2$

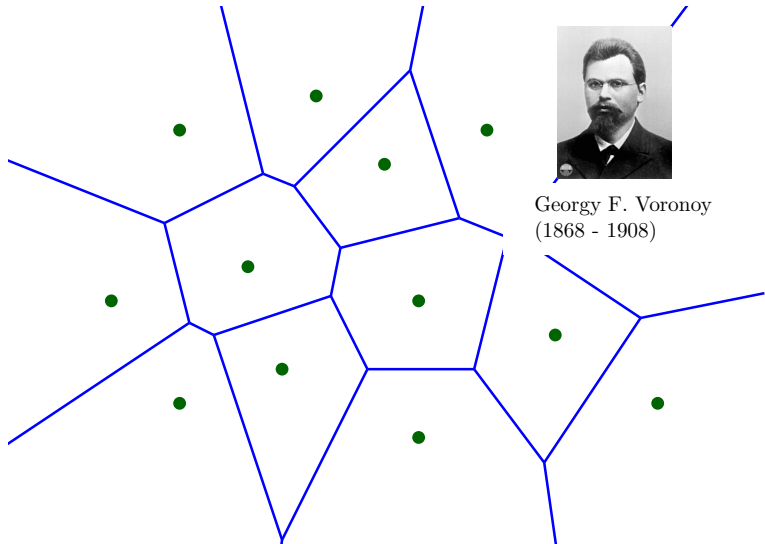
Voronoi cell: $q \in V(p_i) \Leftrightarrow \text{dist}(q, p_i) \leq \text{dist}(q, p_j), \forall p_j \in P, j \neq i$



Example and definition

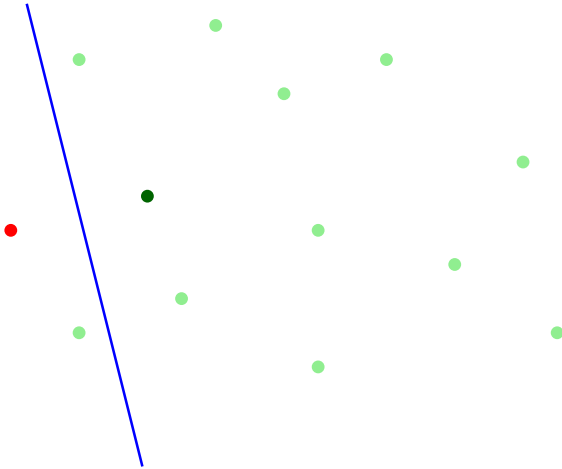
Sites: $P := \{p_1, \dots, p_n\} \subset \mathbb{R}^2$

Voronoi cell: $q \in V(p_i) \Leftrightarrow \text{dist}(q, p_i) \leq \text{dist}(q, p_j), \forall p_j \in P, j \neq i$

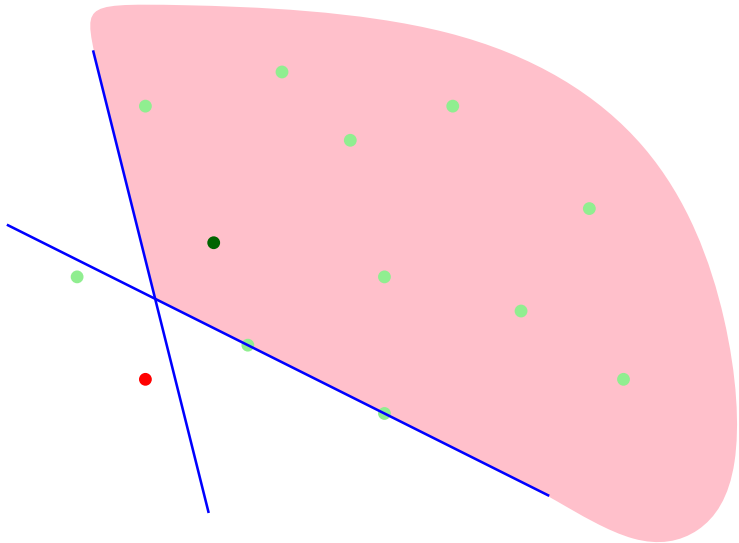


Georgy F. Voronoy
(1868 - 1908)

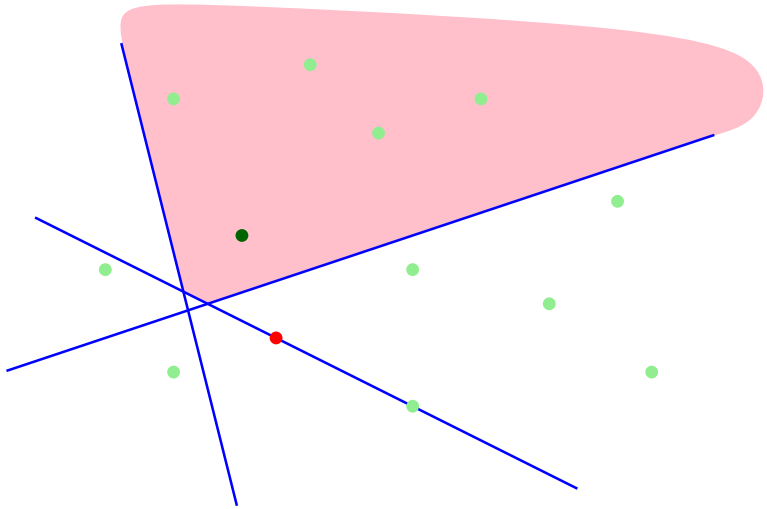
Faces of Voronoi diagram



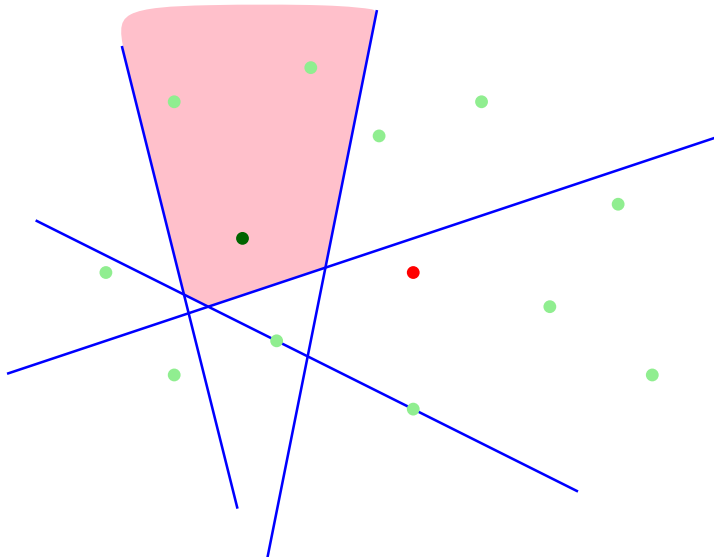
Faces of Voronoi diagram



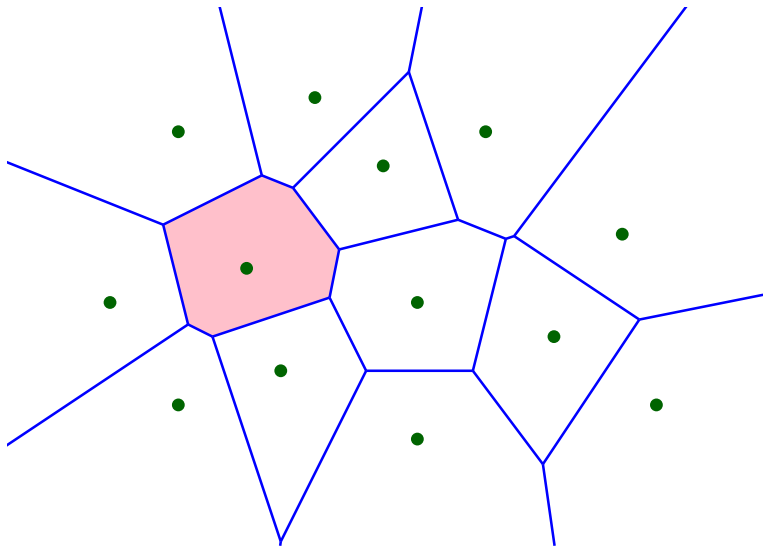
Faces of Voronoi diagram



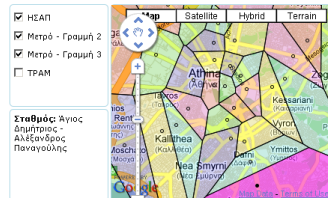
Faces of Voronoi diagram



Faces of Voronoi diagram



Voronoi diagram



Formalization

- sites: points $P = \{p_1, \dots, p_n\} \subset \mathbb{R}^2$.
- Voronoi cell/region $V(p_i)$ of site p_i :

$$q \in V(p_i) \Leftrightarrow \text{dist}(q, p_i) \leq \text{dist}(q, p_j), \forall p_j \in P, j \neq i.$$

- Voronoi edge is the common boundary of two adjacent cells.
 - Voronoi vertex is the common boundary of 3 adjacent cells, or the intersection of ≥ 2 (hence ≥ 3) Voronoi edges.
- Generically, of exactly 3 Voronoi edges.

Voronoi diagram of P = dual of Delaunay triangulation of P .

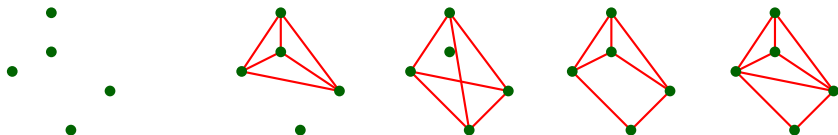
- Voronoi cell $\leftrightarrow$ vertex of Delaunay triangles: site
- neighboring cells (Voronoi edge) $\leftrightarrow$ Delaunay edge, defined by corresponding sites (line of Voronoi edge $\perp$ line of Delaunay edge)
- Voronoi vertex $\leftrightarrow$ Delaunay triangle.

Outline

Triangulation

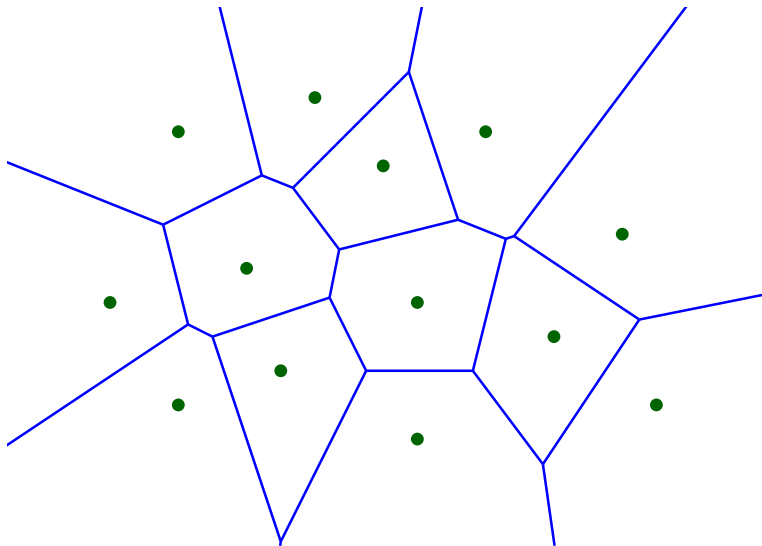
A **triangulation** of a pointset (sites) $P \subset \mathbb{R}^2$ is a collection of triplets from P , namely **triangles**, s.t.

- ▶ The union of the triangles covers the convex hull of P .
- ▶ Every pair of triangles intersect at a (possibly empty) common face ($\emptyset$, vertex, edge).
- ▶ Usually (CGAL): Set of triangle vertices = P .

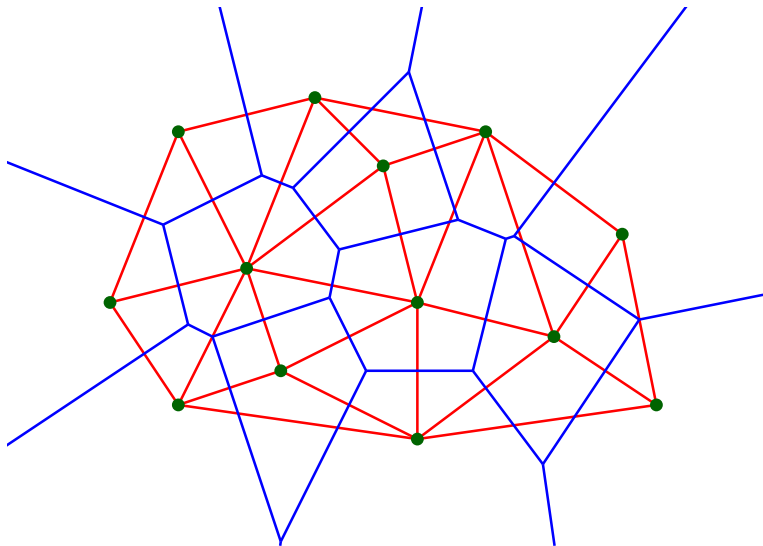


Example: P , invalid, generalized, subdivision, triangulation.

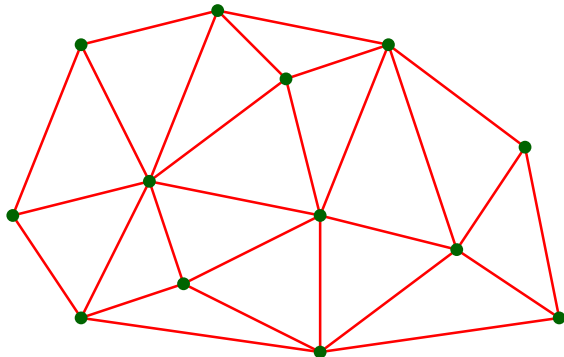
Delaunay Triangulation: dual of Voronoi diagram



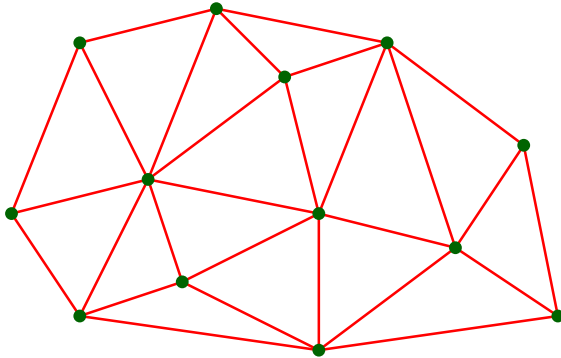
Delaunay Triangulation: dual of Voronoi diagram



Delaunay Triangulation: dual of Voronoi diagram



Delaunay Triangulation: dual of Voronoi diagram

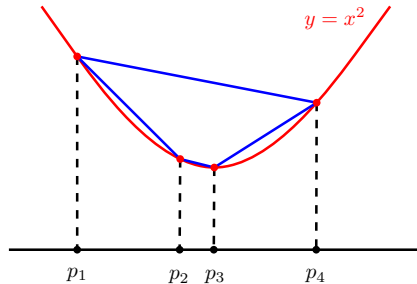


Boris N. Delaunay
(1890 - 1980)

Delaunay triangulation: projection from parabola

Definition/Construction of Delaunay triangulation:

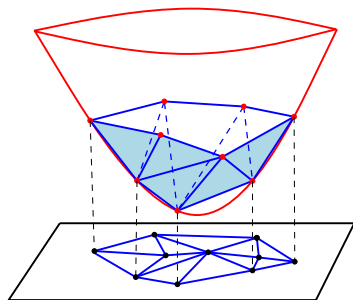
- ▶ Lift input points $p = (x) \in \mathbb{R}$ to $\hat{p} = (x, x^2) \in \mathbb{R}^2$
- ▶ Compute the convex hull of the lifted points
- ▶ Project the lower hull to $\mathbb{R}$



Delaunay triangulation: going a bit higher...

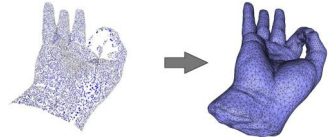
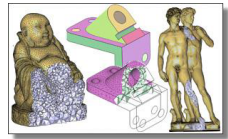
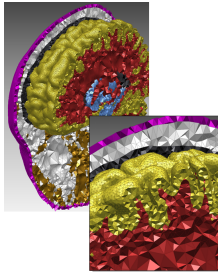
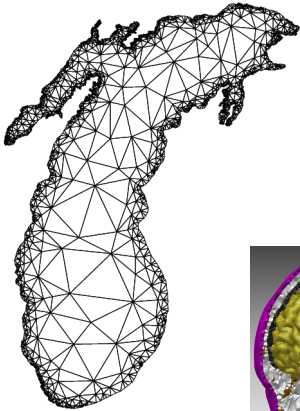
Definition/Construction of Delaunay triangulation:

- ▶ Lift input points $p = (x, y) \in \mathbb{R}^2$ to $\hat{p} = (x, x^2 + y^2) \in \mathbb{R}^3$
- ▶ Compute the convex hull of the lifted points
- ▶ Project the lower hull to $\mathbb{R}^2$: arbitrarily triangulate lower facets that are polygons (not triangles)

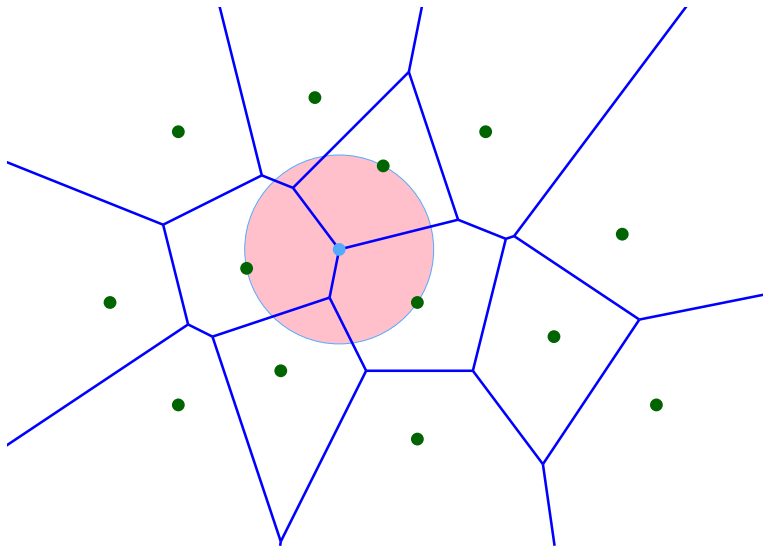


Applications

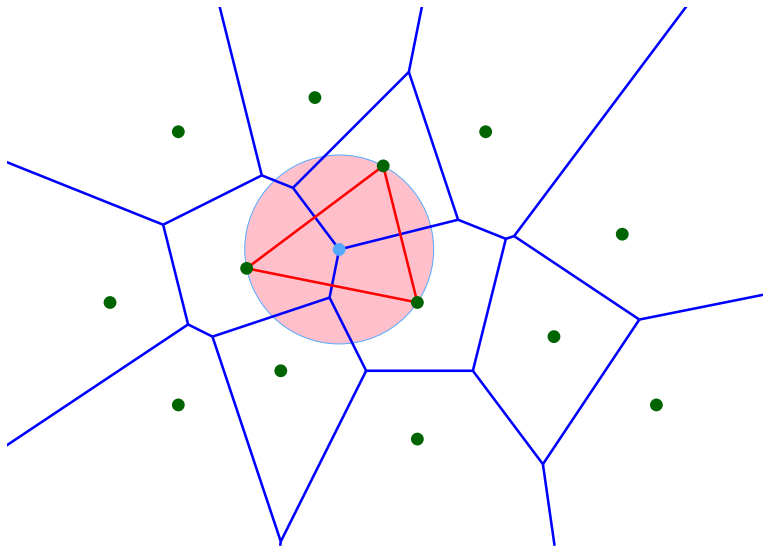
Nearest Neighbors
Reconstruction
Meshing



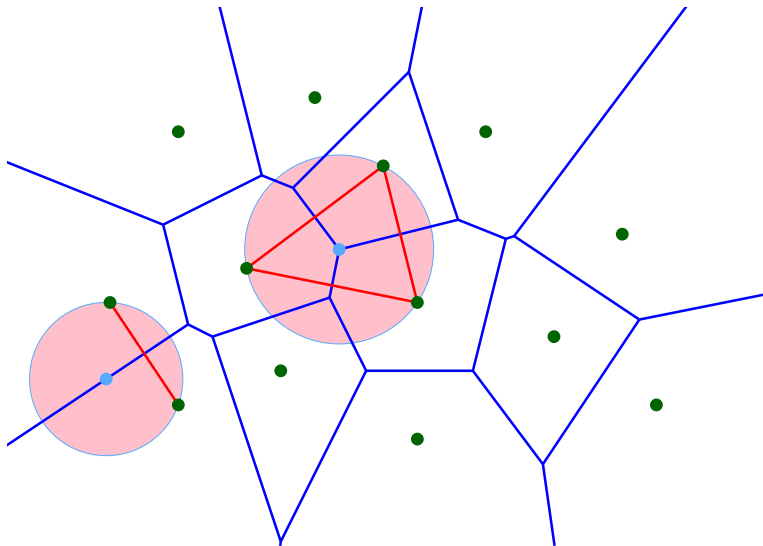
Main Delaunay property: empty sphere



Main Delaunay property: empty sphere



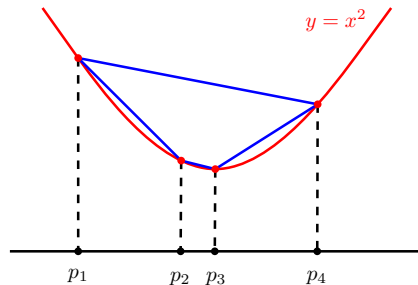
Main Delaunay property: empty sphere



Main Delaunay property: 1 picture proof

Thm (in $\mathbb{R}$): $S(p_1, p_2)$ is a Delaunay segment $\Leftrightarrow$ its interior contains no p_i .

Proof. Delaunay segment $\Leftrightarrow (\hat{p}_1, \hat{p}_2)$ edge of the Lower Hull
 $\Leftrightarrow$ no $\hat{p}_i$ "below" $(\hat{p}_1, \hat{p}_2)$ on the parabola
 $\Leftrightarrow$ no p_i inside the segment (p_1, p_2) .

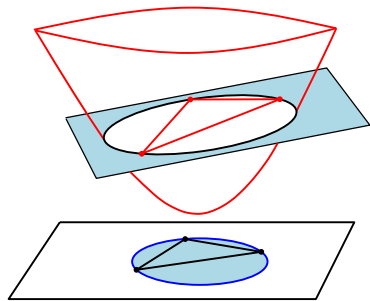


Main Delaunay property: 1 picture proof

Thm (in $\mathbb{R}^2$): $T(p_1, p_2, p_3)$ is a Delaunay triangle $\Leftrightarrow$ the interior of the circle through p_1, p_2, p_3 (enclosing circle) contains no p_i .

Proof. Circle(p_1, p_2, p_3) contains no p_i in interior
 $\Leftrightarrow$ plane of lifted $\hat{p}_1, \hat{p}_2, \hat{p}_3$ leaves all lifted $\hat{p}_i$ on same halfspace
 $\Leftrightarrow$ CCW($\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_i$) of same sign for all i .

Suffices to prove: p_i lies on Circle(p_1, p_2, p_3)
 $\Leftrightarrow \hat{p}_i$ lies on plane of $\hat{p}_1, \hat{p}_2, \hat{p}_3 \Leftrightarrow \text{CCW}(\hat{p}_1, \hat{p}_2, \hat{p}_3, \hat{p}_i) = 0$.



Predicate InCircle

Given points $p, q, r, s \in \mathbb{R}^2$, point $s = (s_x, s_y)$ lies inside the circle through $p, q, r \Leftrightarrow$

$$\det \begin{pmatrix} p_x & p_y & p_x^2 + p_y^2 & 1 \\ q_x & q_y & q_x^2 + q_y^2 & 1 \\ r_x & r_y & r_x^2 + r_y^2 & 1 \\ s_x & s_y & s_x^2 + s_y^2 & 1 \end{pmatrix} > 0,$$

assuming p, q, r in clockwise order (otherwise $\det < 0$).

Lemma. $\text{InCircle}(p, q, r, s) = 0 \Leftrightarrow \exists$ circle through p, q, r, s .

Proof. $\text{InCircle}(p, q, r, s) = 0 \Leftrightarrow \text{CCW}(\hat{p}, \hat{q}, \hat{r}, \hat{s}) = 0$

Delaunay faces

Theorem. Let P be a set of sites $\in \mathbb{R}^2$:

- (i) Sites $p_i, p_j, p_k \in P$ are vertices of a Delaunay triangle $\Leftrightarrow$ the circle through p_i, p_j, p_k contains no site of P in its interior.
- (ii) Sites $p_i, p_j \in P$ form an edge of the Delaunay triangulation $\Leftrightarrow$ there is a closed disc C that contains p_i, p_j on its boundary and does not contain any other site of P .

Triangulations of planar pointsets

Thm. Let P be set of n points in $\mathbb{R}^2$, not all colinear, $k = \# \text{points on boundary of } \text{CH}(P)$. Any triangulation of P has $2n - 2 - k$ triangles and $3n - 3 - k$ edges.

Proof.

- ▶ f : #facets (except ∞)
- ▶ e : #edges
- ▶ n : #vertices

1. Euler: $n - e + f - 1 = 1$; for d -polytope: $\sum_{i=0}^d f_i = 1$
2. Planar triangulation: $3f + k = 2e$.

Properties of Voronoi diagram

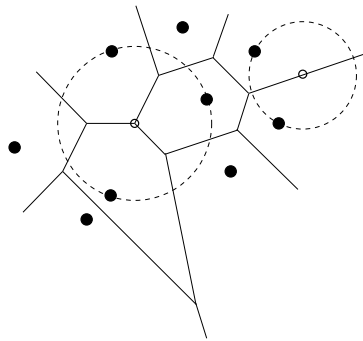
Lemma. $|V| \leq 2n - 5$, $|E| \leq 3n - 6$, $n = |P|$,

by Euler's theorem for planar graphs: $|V| - |E| + n - 1 = 1$.

Max Empty Circle $C_P(q)$ centered at q : no interior site $p_i \in P$.

Lem: $q \in \mathbb{R}^2$ is Voronoi vertex $\Leftrightarrow C(q)$ has ≥ 3 sites on perimeter

Any perpendicular bisector of segment (p_i, p_j) defines a Voronoi edge
edge $\Leftrightarrow \exists q$ on bisector s.t. $C(q)$ has only p_i, p_j on perimeter

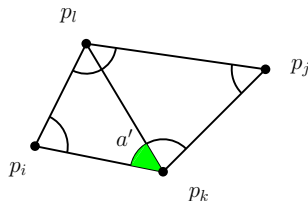
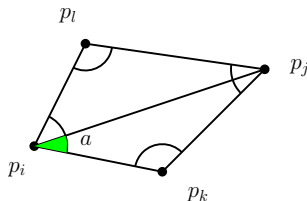


Delaunay maximizes the smallest angle

Let T be a triangulation with m triangles.

Sort the $3m$ angles: $a_1 \leq a_2 \leq \dots \leq a_{3m}$. $T_a := \{a_1, a_2, \dots, a_{3m}\}$.

Edge $e = (p_i, p_j)$ is **illegal** $\Leftrightarrow \min_{1 \leq i \leq 6} a_i < \min_{1 \leq i \leq 6} a'_i$.

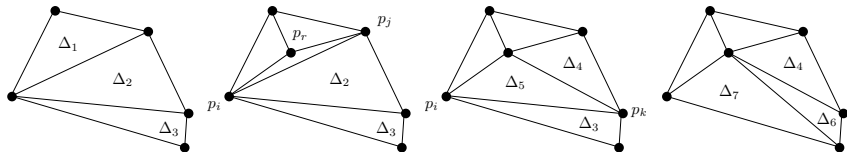


T' obtained from T by **flipping** illegal e , then $T'_a >_{lex} T_a$.

Flips yield triangulation without illegal edges.

The **algorithm terminates** (angles decrease), but is slow.

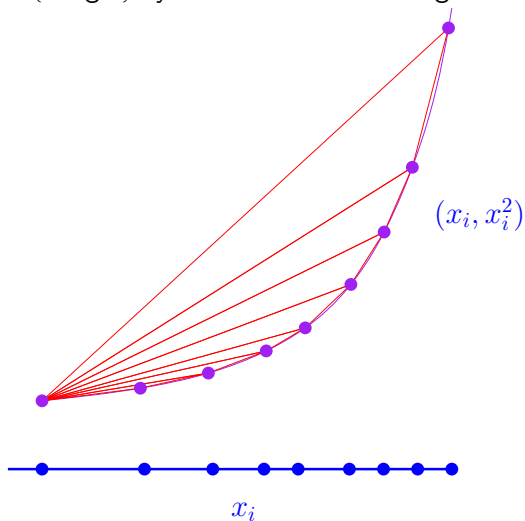
Insertion by flips



Outline

Lower bound

$\Omega(n \log n)$ by reduction from sorting



Delaunay triangulation

Theorem. Let P be a set of points $\in \mathbb{R}^2$. A triangulation $\mathcal{T}$ of P has no illegal edge $\Leftrightarrow \mathcal{T}$ is a Delaunay triangulation of P .

Cor. Constructing the Delaunay triangulation is a fast (optimal) way of maximizing the min angle.

Algorithms in $\mathbb{R}^2$.

- Lift, CH3, project the lower hull: $O(n \log n)$
- Incremental algorithm: $O(n \log n)$ exp., $O(n^2)$ worst
- Voronoi diagram (Fortune's sweep): $O(n \log n)$
- Divide + Conquer: $O(n \log n)$

Voronoi by Lift & Project

Lifting:

- Consider the paraboloid $x_3 = x_1^2 + x_2^2 + x_3^2$.
- For every site p , consider its **lifted** image $\hat{p}$ on the parabola.
- Given $\hat{p}$, $\exists$ unique **(hyper)plane** tangent to the parabola at $\hat{p}$.

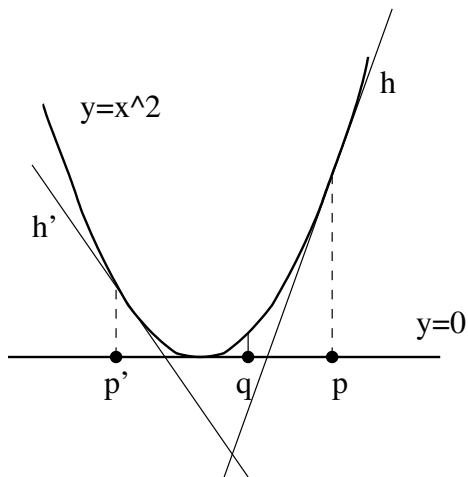
Project:

- For every (hyper)plane, consider the **halfspace** above.
- The **intersection** of halfspaces is a (unbounded) convex polytope
- Its **Lower Hull** projects bijectively to the Voronoi diagram.

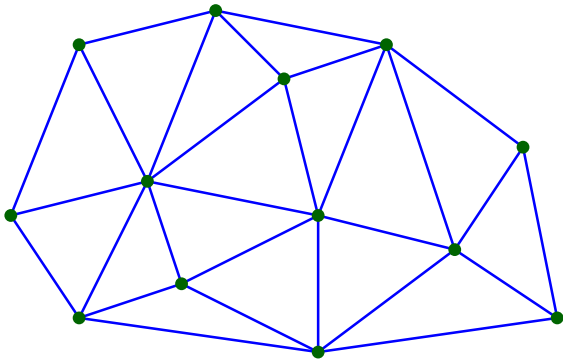
Proof:

- Let $E : x_1^2 + x_2^2 - x_3 = 0$ be the paraboloid equation.
- $\nabla E(a) = \left(\frac{\partial E}{\partial x_1}, \frac{\partial E}{\partial x_2}, \frac{\partial E}{\partial x_3} \right)_a = (2a_1, 2a_2, -1)$.
- Point $x \in$ plane $h(x) \Leftrightarrow (x - a) \cdot \nabla E(a) = 0 \Leftrightarrow 2a_1(x_1 - a_1) + 2a_2(x_2 - a_2) - (x_3 - a_3) = 0$, which is h 's equation.

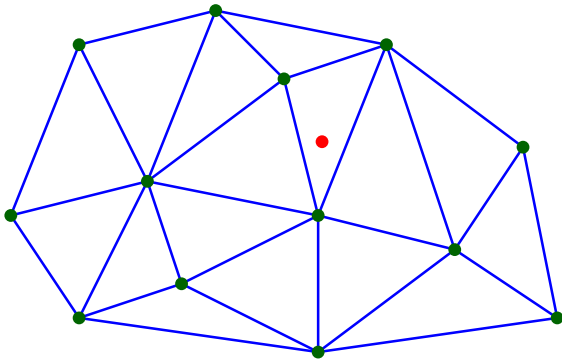
Lift & Project



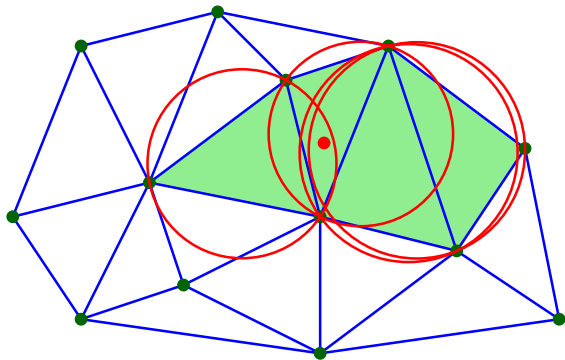
Incremental Delaunay



Incremental Delaunay

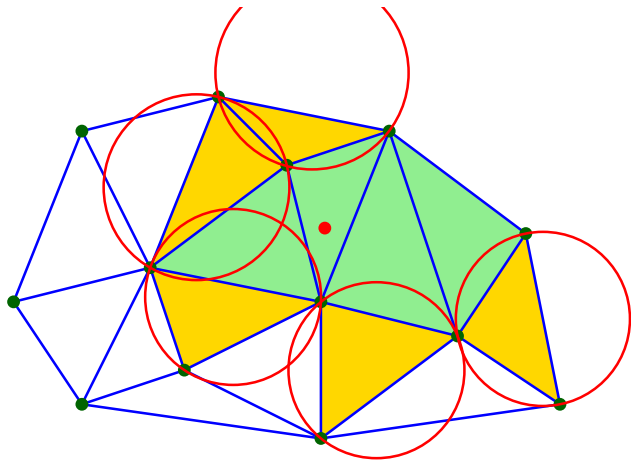


Incremental Delaunay

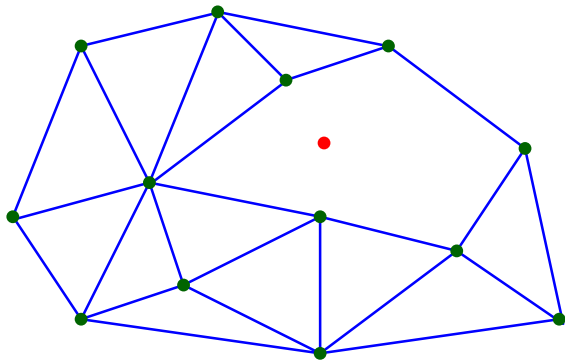


Find triangles in conflict

Incremental Delaunay

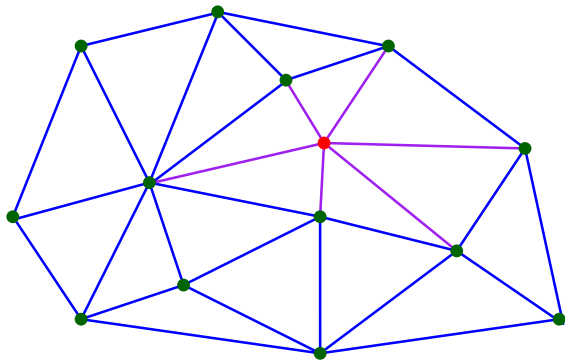


Incremental Delaunay



Delete triangles in conflict

Incremental Delaunay



Triangulate hole

Outline

General dimension polytopes

Faces of a polytope are polytopes forming its extreme elements.

A **facet** of a d -dimensional polytope is $(d - 1)$ -dimensional face:

- The facets of a segment are vertices (0-faces).
- The facets of a polygon are edges (1-faces)
- The facets of a 3-polyhedron are polygons.
- The facets of a 4d polytope are 3d polytopes.

General dimension triangulation

A **triangulation** of a pointset (sites) $P \subset \mathbb{R}^d$ is a collection of $(d + 1)$ -tuples from P , namely **simplices**, s.t.

- ▶ The union of the simplices covers the convex hull of P .
- ▶ Every pair of simplices intersect at a (possibly empty) common face.
- ▶ Usually: Set of simplex vertices = P .
- ▶ Delaunay: no site lies in the circum-hypersphere inscribing any simplex of the triangulation.

In 3d, two simplices may intersect at: $\emptyset$, vertex, edge, facet.

The triangulation is **unique** for generic inputs, i.e. no $d + 2$ sites lie on same hypersphere, i.e. every $d + 1$ sites define unique simplex.

A Delaunay **facet** belongs to: exactly one simplex iff it belongs to $\text{CH}(P)$, otherwise belongs to exactly two (neighboring) simplices.

Complexity in general dimension

- ▶ Delaunay triangulation in $\mathbb{R}^d \simeq$ **convex hull** in $\mathbb{R}^{d+1}$.
- ▶ Convex Hull of n points in $\mathbb{R}^d$ is $\Theta(n \log n + n^{\lfloor d/2 \rfloor})$
Hence d -Del = $\Theta(n \log n + n^{\lceil d/2 \rceil})$
- ▶ Lower bound **[McMullen]** on space Complexity
- ▶ **optimal** deterministic **[Chazelle]**, randomized **[Seidel]** algorithms

Optimal algorithms by lift/project: $\mathbb{R}^2$: $\Theta(n \log n)$, $\mathbb{R}^3$: $\Theta(n^2)$.

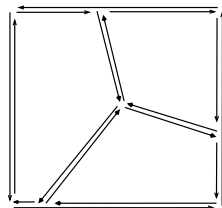
Generalized constructions

In $\mathbb{R}^2$: Various **geometric graphs** defined on P are subgraphs of $\mathcal{DT}(P)$, e.g. **Euclidean minimum spanning tree** (EMST) of P .

Delaunay triangulation $\mathcal{DT}(P)$ of pointset $P \subset \mathbb{R}^d$: triangulation s.t. no site in P lies in the hypersphere inscribing any simplex of $\mathcal{DT}(P)$.

- ▶ $\mathcal{DT}(P)$ contains d -dimensional simplices.
- ▶ hypersphere = circum-hypersphere of simplex.
- ▶ $\mathcal{DT}(P)$ is **unique for generic** inputs, i.e. no $d + 2$ sites lie on the same hypersphere, i.e. every $d + 1$ sites define unique Delaunay “triangle”.
- ▶ $\mathbb{R}^d$: Delaunay facet belongs to **exactly one** simplex $\Leftrightarrow$ belongs to $\text{CH}(P)$

Plane Decomposition Representation



- **Doubly Connected Edge List (DCEL)**
 - stores: vertices, edges and cells (faces);
 - for every (undirected) edge: 2 twins (directed) **half-edges**.
- Space complexity: $O(|V| + |E| + n)$,
 $|V| = \# \text{vertices}$, $|E| = \# \text{edges}$, $n = \# \text{input sites}$.
 - v : $O(1)$: coordinates, pointer to half-edge where v is starting.
 - half-e $O(1)$: start v , right cell, pointer next/previous/twin half-e
- DCEL operations:
 - Given cell c , edge $e \subset c$, find (neighboring) cell c' : $e \subset c'$: $O(1)$
 - Given cell, print every edge of cell: $O(|E|)$.
 - Given vertex v find all incident edges: $O(\# \text{neighbors})$.