15ο ΜΑΘΗΜΑ

05-12-14

1ο παράδειγμα

$$\vec{y}' = A y(t)$$

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

· ιδιοτιμές

$$\det(A - \lambda \mathbb{I}_{n=3}) = 0$$

$$(1-\lambda)^2 (2-\lambda) = 0$$

· $\lambda_1 = 1$, $r_1 = 2$, $d_1 = n - \text{rank}(A - \lambda_1 \mathbb{I}_3) = 3 - 2 = 1$

$$A - \lambda_1 \mathbb{I}_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$ (2 γραμμ. ανεξ. γραμμές)

από ιδιοδιάνυσμα

$$(A - \lambda_1 \mathbb{I}_3)\vec{u} = 0$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$u_2 = 0$

$u_3 = 0$

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$\vec{\varphi}_1(t) = e^t \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Γενικευμένο ιδιοδιάνυσμα 2ης τάξης ($r_1 = 2$, $d_1 = 1$, $r_1 \neq d_1$)

$$(A - \lambda_1 \mathbb{I}_3)^2 \vec{v}_1 = \vec{0} \quad \& \quad (A - \lambda_1 \mathbb{I}_3)\vec{v}_1 \neq \vec{0}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$w_3 = 0$

$$\vec{v}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$\vec{\varphi}_1(t) = e^t(\vec{V}_1 + t(A - \lambda_1 \mathbb{I}_3)\vec{V}_1) =$

(λύση που προκύπτει από το γενικευμένο ιδιοδιάνυσμα)

$\begin{pmatrix} te^t \\ e^t \\ 0 \end{pmatrix} \Rightarrow \vec{\varphi}_1(t) = \begin{pmatrix} te^t \\ e^t \\ 0 \end{pmatrix}$

$(\vec{\varphi}_3(t))$

$\lambda_2 = 2, \quad \tau_2 = 1 = d_2$

$(A - \lambda_2 \mathbb{I}_3)\vec{u}_2 = \vec{0}$

$$\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{array}{l} -x_1 + x_2 = 0 \\ -x_2 = 0 \\ x_3 \text{ αυθαίρετο} \end{array}\right\} \Rightarrow \begin{array}{l} x_1 = 0 \\ x_2 = 0 \\ x_3 \text{ αυθαίρετο} \end{array}$$

$$\vec{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rightsquigarrow \vec{\varphi}_2(t) = e^{2t}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

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2ο ΠΑΡΑΔΕΙΓΜΑ

$\vec{y}'(t) = A y(t)$

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{pmatrix}$$

· ιδιοτιμές

$\det(A - \lambda \mathbb{I}_3) = 0$

$\lambda = 2, \quad \tau = 3, \quad d = 3 - \operatorname{rank}(A - \lambda \mathbb{I}_3) = 3 - 2 = 1$

από ιδιοδιάνυσμα

$$\vec{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rightsquigarrow \vec{\varphi}_1(t) = e^{2t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

· γενικ. ιδιοδ. $2^{ης}$ τάξης

$$(A-\lambda \mathbb{I}_3)^2 \vec{u}_2 = \vec{0} \quad \& \quad (A-\lambda \mathbb{I}_3)\vec{u}_2 \neq 0$$

$$\vec{u}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

$$\vec{\varphi}_2(t) = e^{2t}\left(\vec{u}_2 + t(A-2\mathbb{I}_3)\vec{u}_2\right)$$

· γενικ. ιδιοδ. $3^{ης}$ τάξης

$$(A-2\mathbb{I}_3)^3 \vec{u}_3 = \vec{0} \quad \& \quad (A-2\mathbb{I}_3)^2 \vec{u}_3 = \vec{0}$$

$$\vec{u}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{\varphi}_3(t) = e^{2t}\left(\vec{u}_3 + t(A-2\mathbb{I}_3)\vec{u}_3 + \frac{t^2}{2}(A-2\mathbb{I})^2\vec{u}_3\right)$$

Αν έχουμε το μη ομογενές πρόβλημα

$$\vec{y}'(t) = A\vec{y}(t) + \vec{b}(t)$$

$$\vec{y}(t_0) = \vec{y}_0$$

$$\vec{y}(t) = e^{A(t-t_0)}\vec{y}_0 + \int_{t_0}^{t} e^{A(t-s)} \cdot \vec{b}s \, ds$$

$$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}, \quad \vec{b}(t) = \begin{pmatrix} e^{2t} \\ 0 \end{pmatrix}, \quad t_0 = 0, \quad \vec{y}_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

· ιδιοτιμές

$\lambda_1 = 1, \quad \lambda_2 = 3$

· ιδιοδιανύσματα

$$\vec{u}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

• λύσεις αντιστ. ομογενούς

$$\vec{\varphi}_1(t) = e^{t}\begin{pmatrix}1\\0\end{pmatrix}$$

$$\vec{\varphi}_2(t) = e^{3t}\begin{pmatrix}1\\1\end{pmatrix}$$

• $e^{At} = \Phi(t)\,\Phi^{-1}(0)$, όπου Φ Θ.Π.Λ. της αντιστ. ομογ.

$$\Phi(t) = \begin{pmatrix} e^{t} & e^{3t} \\ 0 & e^{3t}\end{pmatrix}$$

$$\Phi(0) = \begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix} \Rightarrow \Phi^{-1}(0) = \begin{pmatrix} 1 & -1 \\ 0 & 1\end{pmatrix}$$

$$e^{At} = \begin{pmatrix} e^{t} & e^{3t}-e^{t} \\ 0 & e^{3t}\end{pmatrix}$$

$$\vec{y}(t) = \begin{pmatrix} e^{t} & e^{3t}-e^{t} \\ 0 & e^{3t}\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix} + \int_0^t \begin{pmatrix} e^{t-s} & e^{3(t-s)}-e^{t-s} \\ 0 & e^{3(t-s)}\end{pmatrix}\begin{pmatrix}e^{2s}\\0\end{pmatrix}ds$$

$$= \begin{pmatrix} 2e^{3t}-e^{t} \\ 2e^{3t}\end{pmatrix} + \int_0^t \begin{pmatrix} e^{t+s} \\ 0\end{pmatrix}ds = \begin{pmatrix} 2e^{3t}-e^{t} \\ 2e^{3t}\end{pmatrix} + \begin{pmatrix} \int_0^t e^{t+s}\,\cancel{2e^{3t}}\,ds \\ \int_0^t \cancel{2e^{3t}}\, 0\, ds\end{pmatrix}$$

$$= \begin{pmatrix} 2e^{3t}+e^{2t}-2e^{t} \\ 2e^{3t}\end{pmatrix}$$

Ασκήσεις

1) $\vec{y}'(t) = A\,\vec{y}(t)$

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1\end{pmatrix}$$

2) $\vec{y}'(t) = A\vec{y}(t) + \vec{b}(t)$

$\vec{y}(t_0) = \vec{y}_0$

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}, \quad \vec{b}(t) := \begin{pmatrix} 0 \\ 0 \\ e^t \cos 2t \end{pmatrix}, \quad t_0 := 0, \quad \vec{y}_0 := \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$