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Making sense of and with mathematics: the interface between academic mathematics and mathematics in practice

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Abstract This article summarises what we can learn from research into workplace practice and vocational preparation to inform the design of mathematics education curricula for learners in general education undertaking compulsory schooling. Key findings about workplace practices are identified and explicated through the report of a case study from research in which researchers, students, teachers and workers explored workplace and mathematical practices together. Further to this, issues of learning and personal development are considered and explored from a point of view that sees learning as practice (doing) and identity development (becoming). This leads to a proposal for principles that provide a strategic vision for curriculum design. A potential approach to tactical design that facilitates curriculum structuring is illustrated in the particular instance of understanding developing measures as a modelling activity. Overall, the exercise, whilst providing some insight into possible ways forward in curriculum development, also suggests areas that require further research and development.

Keywords Workplace mathematics · Mathematics curricula · Transition · Activity · Boundaries

1 General mathematics education curricula: a twenty-first century design challenge

The issue of the transferability of what is learned and known from one setting to another has greatly troubled educators generally, and the mathematics education community in particular, over many years. The classical cognitivist view might be characterised as suggesting that the generality of mathematics and its abstract nature means that it should be possible to easily apply it in a wide range of situations. On the other hand, situated cognitionists demonstrate how the context of situations is often inextricably bound up with the development, understanding and learning of mathematics for individuals, and indeed communities, in ways that inhibit such transferability (e.g., see the work of Lave, 1988). Others, such as Beach (1999) and Säljö (2003), suggest that the metaphor of *transfer* itself is problematic and that it is not surprising that supporters of these different perspectives conceptualise transfer very differently;

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consequently, it is difficult to compare findings and arguments. What seems clear from research into the use of mathematics in workplace activity, as the complex accounts testify (e.g. Hoyles, Noss, Kent, & Bakker, 2010; Roth, 2012; Triantafillou & Potari, 2010; Williams & Wake, 2007a, b; Zevenbergen, 2011), is that mathematical activity in workplaces, where and when it occurs, looks very different from that in educational settings. Indeed, there is a fundamental difference between the nature of the role of mathematics in the different settings of school and workplace that has a major impact on mathematics as practised in each. In school, mathematics is mainly the object of study, whereas in the workplace, it is used as a tool for mediating activity that is inevitably focused on the productive outcomes that are the *raisons d'être* of the workplace. For individuals, as workers or learners, there are consequently different motivations for, and expected outcomes from, their mathematical actions. Mathematics in workplaces takes on different formulations from those familiar in schools because of the very different role it plays for those who engage with it and also because of the diverse range of technologies (considered in the widest sense) available and used.

Given that mathematics plays an important role throughout the world in compulsory general education, this article seeks to understand what we can learn from studies of mathematics in workplace practice that can inform the design of mathematics curricula for compulsory general education. It is not my intention to argue here for a more utilitarian curriculum that is focused on preparation for work: such a suggestion is more suitable for designers of vocational and pre-vocational courses where progression routes are more clearly defined. Rather, it is my intention to consider what we might distil from our research and emerging understanding of mathematical activities in workplace settings that might inform future iterations of curriculum design. This seems particularly important as this research provides new insights into the role of mathematics and its dual nature as both an area of study and an increasingly diverse tool with application in and across many aspects of human life. At a basic level, workplace research emphasises the dynamic nature of this *tool-use* of mathematics and suggests that we somehow need to capture this in future curriculum manifestations.

In workplaces, as researchers, our attention is drawn to practice, that is, the coupling of human actions with the relatively complex tasks or problems that workers undertake and the role that mathematics has to play in these. In seeking to inform how we might develop a mathematics curriculum for compulsory schooling from research into workplace activity, however, it seems clear that we need to look beyond *particular* manifestations of competence in *specific* practices to a generality of what is emerging as essential across a wide range of such practices. As these practices become increasingly diverse, with many being enmeshed in the use of increasingly sophisticated and changing technologies, it is important that we consider how a twenty-first century mathematics curriculum might best prepare young people to engage during their lifetimes with, and make sense of, such practices as they emerge and develop; in many cases, in directions as yet unknown.

Much recent workplace research has drawn, to a greater or lesser extent, on Cultural Historical Activity Theory (CHAT) (see, e.g., Roth, 2012; Williams & Wake, 2007a) which considers school/college and workplaces as activity systems that are culturally and historically evolved, with the actions of the individual contributing to the activity of the collective community in pursuit of its goals in terms of outcomes. In these terms, the school and the workplace focus mainly on learning and production, respectively, with the former preparing students for progression to the latter in its many different formulations, as well as for progression to adulthood and citizenship more generally.

From this theoretical perspective, one way of providing a bridge between the activity systems of workplace and education, so that purposeful learning might be developed, is to provide for boundary crossing (Engeström, Engeström, & Kärkkäinen, 1995) by students as they become workers. Learning can be mediated in such cases by specially created *boundary objects*—artefacts that are sufficiently robust to maintain meaning across communities, but also flexible enough to be used effectively in each (Star, 1989). Akkerman and Bakker (2011) provide a comprehensive overview of boundary objects and boundary crossing in their review of the literature and identify the potential for learning at boundaries through the mechanisms of identification, coordination, reflection and transformation. Such bridging approaches incorporating boundaries have been explored profitably in relation to the workplace learning of workers (see, e.g., Hoyles et al., 2010) and in the case of students in specific vocational preparation for laboratory work (Bakker & Akkerman, 2013). In their work, Hoyles et al. (2010) designed computer-based learning environments, which they termed *technology-enhanced boundary objects*, that provide opportunities for mathematical insight and learning in particular workplace settings. These have been shown to have efficacy in better preparing workers in the packaging industry to control production processes, in the automotive industry to understand the statistics of process control and in the financial services industry to understand the financial products they sell to customers. Bakker and Akkerman considered the use of reports as boundary objects that had meaning in both vocational school and workplace settings. They provide insight into how students and their workplace supervisors brought together both mathematical (in this case, statistical) and workplace perspectives to develop new understandings of the application of mathematics in practice. In preparation for specific and immediate work, these boundary objects provide for quite a tight coupling between mathematical activity and a specific workplace setting. This approach does not, therefore, provide an immediately obvious solution for the conceptualisation and development of a new epistemology and resulting mathematics curriculum with more general applicability. However, it does seem apparent that boundaries and boundary objects have a potentially significant role to play in informing our consideration of curriculum design.

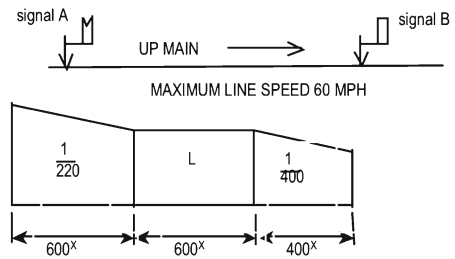
In considering mathematics curricula for general education, we have to seek solutions that are less clearly focused on apparent progression routes; crucially, we need to inform strategic design of a curriculum that seeks to adequately prepare students for future knowledge transformation to settings as yet undetermined. Burkhardt (2009) helpfully identifies three major aspects of educational design—strategic, tactical and technical—with the first being "concerned with the overall structure of the product set and how it will relate to the user-system" (pp. 1–49). In later sections in this article, I focus on emerging issues that might inform principles for strategic design and propose, at a tactical design level, one possible way of providing an organising structure of a mathematics curriculum. The final part of the design process, that of technical design, at the level of detail concerned with products that speak directly to the end users (i.e., teachers and students) is beyond the scope of this article. Prior to putting forward any proposals, I will summarise some of the key issues that analysis of practice identifies as relevant to inform such design, using for illustrative purposes a vignette from my own research. I will also consider perspectives on learning as practice and as identity development.

2 Vignette: the railway signal engineer

This vignette is taken from one of a dozen case studies developed as part of a project which explored workplace practices that involved aspects of some mathematical activity and involved workers, college teachers and their students who were following a relevant pre-vocational course. Each of the studies was developed as a result of about 5 days of fieldwork, during which we, as researchers, observed workers' practices and discussed these with them, prior to undertaking a visit with students, during which we explored the practices in terms of mathematics. These workplace visits were followed up by further visits to the students' college where issues that arose with the students and their teachers were discussed. A cross-case analysis allowed conclusions about the general nature of mathematical activity in workplace practices to be drawn (Wake & Williams, 2001).

The case study summarised here brought together a group of college students (aged 16–19) following a pre-vocational engineering course, their teacher and a researcher to investigate the work of Alan, a railway signal engineer. Elsewhere (e.g., Williams & Wake, 2007a), such data has been analysed using CHAT to understand the actions of individual workers in relation to the activity of the workplace community, but here, I wish to focus closely on the nature of the mathematical understanding at issue. This particular case study was chosen for use here because the mathematics is easily accessible, and the context of the railway is assumed to be relatively familiar to most readers.

During a visit of the group to Alan's workplace, he described and illustrated how he calculates the speed at which trains should travel on a stretch of track. As part of this work, Alan decides where to position the speed boards that indicate to drivers the maximum speed at which they should travel as they head toward an impending signal at which they may have to stop. The gradient of the track is an important factor in the calculations Alan performs: because of the momentum of the train, a downhill gradient will require a greater stopping distance, and so a greater distance between speed board and signal, whilst for uphill gradients, the reverse is true. Consequently, Alan needs to calculate the average gradient over a stretch of track. He showed the students the example from the training manual (see Fig. 1). Having calculated the average gradient for the stretch of track, Alan goes on to use the chart in Fig. 2 to determine the distance between signals and speed boards for trains of varying maximum speed limits. Alan knows when using the chart that, for safety reasons, he should always use the maximum possible speed for the types of trains that will use the track and the gradient which is least advantageous for stopping the train.



SECTION	DISTANCE IN YARDS	GRADIENT	RISE (YDS) (DIST * GRAD)	FALL (YDS) (DIST * GRAD)
X - A	600	1/220	-	2.73
A-B	600	L	-	-
B-Y	400	1/400	-	1.00
TOTALS	1600			3.73

$\frac{1600}{3.73} = 428.95$ THERFORE AVERAGE GRADIENT= $\frac{1}{429}$ FALLING

Fig. 1 Example of average gradient calculations from railway signal engineering training manual (Note: The use of *superscript x*, as in 600^x, is to indicate yards, an imperial unit of distance)

INITIAL SPEED (Mph)	GRADIENT Rising 1 in 50 2.0%	1 in 67 1.5%	1 in 100 1.0%	1 in 200 0.5%	Level Level	Falling 1 in 200 0.5%	1 in 100 1.0%	1 in 67 1.5%	1 in 50 2.0%
20	175	180	195	215	240	275	320	395	520
25	240	255	280	315	355	410	485	625	840
30	320	340	380	425	485	575	700	895	1425
35	405	440	485	550	635	780	1010	1380	2237
40	495	550	620	720	865	1080	1420	1903	2237
45	630	710	805	935	1130	1435	1660	1903	2237
50	688	748	816	935	1130	1435	1660	1903	2237
55	770	831	901	984	1130	1435	1660	1903	2237
60	849	911	980	1061	1165	1435	1660	1903	2237

Fig. 2 Railway signal engineering chart used to determine the distance required between signals and speed boards

It was clear that Alan does not need to think hard about all this; for him, the required understanding has become operationalised (Leont'ev, 1978), or automatic, in his practice. He knows how to find the two values of gradient included in the table between which his calculated gradient lies, and then to use the value from the column in the table "to the right", thus leaving a "margin for error" for safety. Follow-up discussions with the group of students who visited the workplace highlighted the major misunderstandings they had in relation to gradients.

The students first of all investigated how to find the distance between the stop and the warning signals (speed boards) using a gradient of *1 in 433 rising* that appears on a plan of a section of track (Fig. 3).

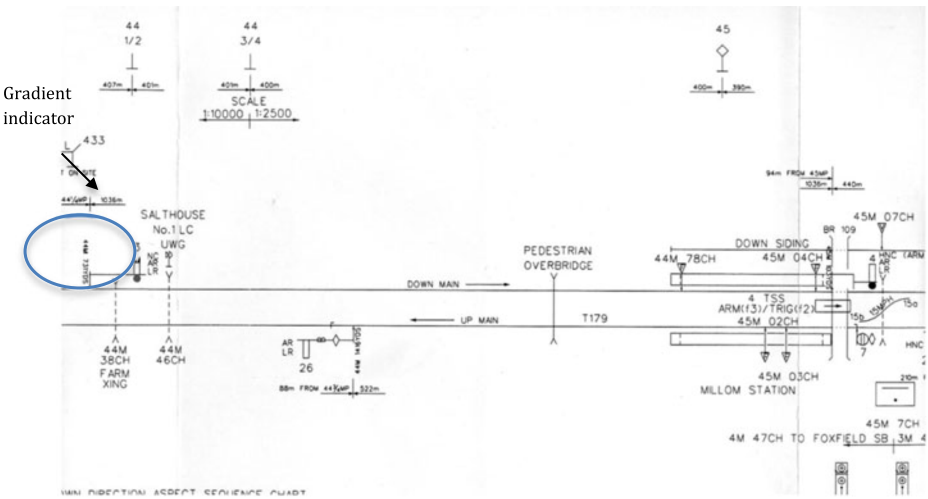


Fig. 3 Railway signal engineering plan of a section of railway track

Student 1: Yeah, or 1 in 433—and it is *rising*—it couldn't go that high could it?

Student 2: I don't know how they use it. [referring to Fig. 3]

Researcher: well look, ok… have you got any idea (to S2). Here you've got *level*, here you've got 1 in 200, and here you've got 1 in 100 [indicating the column headings]. But we want 1 in 400 so what do you think we might do?

S1: double it [indicating value in 1 in 200 column].

S2: yeah double it…

R: …. double the distance would make it *huge* wouldn't it? Yes, if you doubled… like 60 miles [per hour]… we're working along this line here… So 1 in 200 is 1061 if you *doubl*ed it, that would be 2100—well…. but on the level it's only 1100?
… if we could slot in another column there between 1 in 200 and? Yes?

S1: I would just take the higher gradient… or erm…

R: Or you could… oh you see, you could… erm. Ah, he [Alan] mentioned something about always using the one to the *right*, didn't he?…

S1: right yeah

R: so, where are we then? Which… I can't read that. What value have you got?

S1: 1165

R: that's 1165… what are they… they're yards.[1] Right, so what we've got to find out *now*, is… whether it is actually 1165 …

S2: times 2

The students had great difficulty in making sense of what had become operationalised in Alan's practice. As student 2's final comment indicates, his initial belief that a gradient of 1 in 400 somehow requires doubling values associated with a gradient of 1 in 200 seems intuitive and a deeply ingrained misconception common to these students.

Equally problematic was the students' understanding of the training calculations of Fig. 1 that show how to find the average gradient over three of sections of track. Alan explained that such averaging needs to be performed prior to using the chart in Fig. 2. The brief extract below from the transcript of the discussions between researcher and one student from the group illustrates how the student struggled to make sense of the process.

R: Yes… So can you just explain what's going on in there [indicating the Table in Fig. 1]?

S: … used different gradients for each slope and he's averaged it out…

[1] Yards are an imperial measure of distance still used throughout the railway industry and other transport systems in the UK. One yard is a little shorter than 1 m.

R: yes can you sort of explain the detail…

S: you started adding them together—adding the gradients together and divide by two.

R: Perhaps if we describe what each column is doing

Here, the student appears to associate finding an average with the school mathematics procedure of "adding the values together and dividing by the number of values"—presumably, in this case, ignoring the level section of track. The ensuing discussion was lengthy, requiring the researcher to explain the basic concept of gradient by drawing diagrams that illustrated "for every 220 it goes along, it comes down 1, so when you've gone along [the track] 220 it's come down 1". Of course, Alan's familiarity with the procedure caused him no problems with his calculations. However, attempting to understand this for the first time was clearly demanding for the students.

This particular case demonstrates that although the workplace practice is intricately linked with knowledge of the school mathematics concepts of gradients and averages, the demands made of outsiders in making sense of this seem to leave those who have developed their mathematical knowledge, skills and understanding in current general mathematics curricula in school settings ill prepared. This is the case even though there is a well-set-out example to follow in Alan's manual, with considerable contextual information and background detail (that may or may not help the outsider). The transcript of the students struggling to make sense of the training example points to fundamental problems in their understanding of gradient and averages and consequently how to calculate the average of two or more gradients. Throughout the process, they attempted to rely on procedural approaches of which they clearly lacked any deep understanding.

3 Vignette commentary

This particular vignette serves to highlight many of the key features of workers' activity and their mathematical understanding when contrasted with the problems that surfaced for the students in the new workplace/college interface activity system that the research initiated. The complexity of workplaces is immediately noticeable, both in terms of social structuring and the activities in which the particular workers we researched were engaged. This often leads to the mathematics being made invisible or *black-boxed* (Williams & Wake, 2007a), at times deliberately so. Table 1 summarises the key findings that emerged across the entirety of our case studies (Wake & Williams, 2001) and identifies how the vignette here adds illustrative texture to these.

In general, and perhaps obviously so, workers do not consider that they are doing mathematics (Wake & Williams, 2001; Williams & Wake, 2007b): their goal-directed activity (Leont'ev, 1974) is focused on workplace production, and in our project, it was the research that provided the catalyst for identification and unpacking of the mathematics, in effect provoking a breakdown (Pozzi, Noss, & Hoyles, 1998) of the routine. This hiding of mathematics, at least as far as workers are concerned, tends to occur through its black-boxing. Black-boxing of mathematics in this way often has the deliberate intention of minimising the level of scientific mathematical knowledge with which workers are required to engage; either so that mistakes do not result from incorrect mathematical working or because of the complexity of the process that the mathematics underpins. In many cases, the mathematics itself is relatively impenetrable to the majority of workers, with the black-boxing ensuring that

Table 1 Major findings in relation the to use of mathematics in workplace practice exemplified in relation to the railway signal engineer vignette

Key finding	Vignette observation
In workplaces, knowledge is often crystallised (Hutchins, 1995) in artefacts, including tools and signs, often as a result of reification by workplace communities (Wenger, 1998).	The chart in Fig. 2 is just one of a number used by the railway signal engineer that effectively crystallises knowledge and understanding of the use of equations of motion and consideration of forces. In his interview, the engineer referred to issues of knowledge and understanding but also explained that in his day-to-day practice, this is made unnecessary because of artefacts such as this chart.
Use of mathematics is often "black-boxed" (Williams & Wake, 2007a) and engagement with mathematics often only occurs at "breakdown" moments (Pozzi et al., 1998).	The chart in Fig. 2 effectively black-boxes the need for more advanced mathematical knowledge. Workers do not need to engage with concepts of forces and acceleration. This might be provoked if, for example, there was a request for specific values intermediate to those given in the chart or if a newcomer seeks to understand how the chart is developed (as was the case during our research).
The fusion (Meira, 1998) of mathematical "signs", in the sense of Peirce (1931–1958) with the reality they represent reduces cognitive effort.	Consider, for example, the mathematical sign highlighted in Fig. 3. This indicates that to the left of this point, the gradient of the track is level, and to the right, it is 1 in 433 rising. This symbol/sign is particularly meaningful to the railway signal engineer, whereas it requires cognitive effort by "outsiders" in its interpretation and eventual use.
When required, mathematical understanding is only part of a complex interconnected set of conceptual resources on which workers have to draw as they engage in often complex and substantial activity which builds on, and involves, other workplace colleagues' knowledge, skills and understanding.	The particular example of the railway signal engineer as presented in the vignette here is particularly well focused on mathematical activity as practised by one worker. However, he explained that, in deciding exactly where to place the speed boards, he relies on additional contextual information, for example in relation to line of sight, which he might have to ascertain from a colleague familiar with the particular section of track at issue. The plans of the rail network which the worker uses are stylised and idiosyncratic (see below) and are not conventional (or at least widely used) straightforward maps.
Workplaces present outsiders with a diversity of conventions and idiosyncrasies in both representation and methods of analysis: these are almost always quite different from those adopted in academic mathematics, provoking breakdowns (Pozzi et al., 1998), problem solving and modelling (Wake, 2013).	The work of the railway signal engineer relies on a large number of artefacts, such as Fig. 3, that use industry-specific conventions and notations to represent the reality of the railway track. Again, consider, for example, the highlighted symbol that indicates a change in gradient and contrast this with how gradient would be represented in a school mathematics text.
Workplace activity with mathematics as central often relies on relatively simple mathematics embedded in complex situations (Steen, 1990). Making sense of this also provokes breakdowns, problem solving and modelling.	In general the concept of gradient might be considered relatively straightforward. Here, it is given complexity by its use over a number of sections of track, and the need to be able to calculate an overall (average) gradient.

workplace outcomes are routinised and error free. This, however, does not obviate the need for all mathematical thought, as the black-box itself produces output in relation to input, which needs interpretation and understanding. It is this sort of output that, for experienced workers, becomes *fused* (Meira, 1998) with workplace contextual factors. For example, workers often interpret graphical output not in terms of controlled and independent variables, or statistical measures of location and spread, but instead talk of these representations or *signs*, in the sense of Peirce (1931–1958), in terms of the workplace features they re-present. In one of our case studies, for example, a chemical engineer explained a graph, made complex due to its dual vertical axes and logarithmic scales, in terms of the chemical reaction it represented (Williams, Wake, & Boreham, 2001). It is when newcomers to the workplace attempt to understand established practice, or when such practice breaks down because changes are introduced (Pozzi et al., 1998), that workers often need mathematical knowledge, skills and understanding to (re-)construct meaning of what, by necessity, becomes routine in day-to-day practice.

The vignette reported here identifies typical manifestations of activity underpinned by mathematics in workplaces. This presents us with the conundrum of how we might draw on our developing understanding of workplace activity such as this to inform directions for curriculum development that might lead usefully to the preparation of young people as potential future workers in such situations. As signalled in the first section of this article, many researchers and theorists in studies that have probed the use of mathematics in workplaces, and the difficulties that workers encounter, have indicated that the notion of transfer is at the heart of the problem.

In the next sections, I wish to explore the nature of learning more widely than its usual conception in schooling, which I suggest is often narrow and focused on a vertical view of cognitive development, in which knowledge becomes increasingly abstract and removed from everyday experience. In doing so, I draw on a number of theoretical ideas that typically emerge from studies into mathematically based workplace activity and where they have been applied as analytical tools. As Tuomi-Gröhn, Engeström, and Young (2003) point out in the introduction to their edited volume that explores transfer and boundary crossing between school and work, many researchers in this field take the view that transfer incorporates polycontextuality, that is, simultaneous participation in, and boundary crossing between, multiple communities of practice. Their collection considers vocational learning and work, whereas I aim to consider general mathematics education, which, by its nature, is less clearly focused on the specifics and particularities of a given sector of employment. However, in the next section, I propose a way forward by taking a similar stance on transfer to researchers such as Beach (1999). Tuomi-Gröhn et al. summarise this approach as, rather than focussing on knowledge, instead focus should be on a process of generalisation that is located in the changing relationships between individuals and sets of social activities. Further, these relationships are embodied in systems of symbolic artefacts that are created to facilitate human actions in different settings. Thus, artefacts that have meaning across different settings, albeit different meanings in each, appear to have particular importance because of the boundary crossing and learning they can potentially stimulate. I then explore theoretical conceptualisation of issues of learning across social settings before synthesising these ideas to propose some general principles for curriculum design at a strategic level.

4 Horizontal and vertical conceptions of learning and personal development

The issues of *situativity* and the social, ideas of *horizontal* and *vertical* development of the person, both in relation to knowledge and to their identity, and related consideration of individual and societal values, all seem to have important relevance, but require some careful unpacking.

In terms of knowledge development, Vygotsky recognised the potential of spontaneous conceptualisation arising from reflection on everyday experience, as opposed to nonspontaneous or scientific conceptualisation that comes from beyond this, and as part of a system of interdependent concepts (formalised academic knowledge). Vygotsky (1986) suggested that "…the development of the child's spontaneous concepts proceeds upward, and the development of his scientific concepts downward…" (p. 193).

We might, therefore, consider an individual's everyday understanding as emerging from horizontal activity through interaction with the world within a number of different communities and, in contrast to this, the vertical development of knowledge as relating to formal structured educational activity. In mathematics education, these ideas are reflected in those of horizontal and vertical mathematising, as developed by Treffers (1987) in theorising the essence of Freudenthal's Realistic Mathematics Education (RME). From this perspective, horizontal mathematising is about moving *horizontally* back and forth between situations or contexts in which mathematics arises or is applied. In the RME approach, *realisable* situations and contexts outside of mathematics (the home, school or workplace, but culture generally) provide a resource for mathematising. Thus, context provides the catalyst for developing mathematical representations which can then, as objects of study, be developed further, vertically, for use in future horizontal mathematisation. Typically, the relation between the model that was initially used to organise a context (say, the number line used to represent whole numbers in a context involving adding and subtracting positive whole numbers) is then used to extend the mathematics with which the learner is familiar. For example, the properties of the number line are used to discover new properties of the numbers or even new numbers (Gravemeijer, 1994, 1998). First, the model has been introduced as an organising model *of* the context, but then, the number line becomes a model *for* the mathematical objects themselves, allowing the learner to invent/develop/discover new mathematics quite intuitively. In the main, such approaches have used realisable contexts for horizontal mathematisation, where the context need not be real but rather imaginable to the learner. It would seem important in adapting such a model of curriculum design that more suitably reflects future workplace practice and that the clutter and messiness of such reality forms an integral part of the problem situation that requires mathematising. In their work in educational design, Dierdorp, Bakker, Eijkelhof, and van Maanen (2011) researched the use of authentic workplace practices as the basis for materials that organised hypothetical learning trajectories in developing students' informal inferential learning. The successful outcomes of this research in terms of student motivation and learning suggest that such a methodology has the potential to inform curriculum design throughout strategic, tactical and technical levels.

There appear to be parallels in discussion of the horizontal and vertical development of knowledge and understanding generally, and mathematics more particularly, with the use of these terms in discussion of transformation, or development, of the person. In considering development of the individual in terms of their learning, Beach's (1999) construct of *consequential transitions* appears particularly helpful. In his critique of the usual understanding of transfer—that is, the abstraction of knowledge and understanding (particularly pertinent in the case of mathematics) and its transportation and application to a range of settings—Beach considers how it might be better conceptualised as the (re-)construction of knowledge, skills and identity in relation to different communities of practice (Wenger, 1998). Wenger's development of ideas of community of practice I consider here as providing a lens, complementary to CHAT, through which to view the co-production of participants of workplace and school knowledge. This provides a perspective that emphasises participation and identity formation of individuals, in contrast to a focus on how the system is constituted in pursuit of a common goal, as emphasised by CHAT. Further to this, Beach asks us to expand our notion of developmental progress beyond the usual vertical (in Beach's terms, *lateral*) transitions as we move in what constitutes socially accepted developmental progress between historically related activities: for example, from college to university or work or from university to work. In doing so, he asks us to consider, as equally valid, developmental progress that he terms *collateral* transitions, which involve, for example, simultaneous participation in both workplaces and education or training. In his research, Beach contrasted shopkeepers and students participating in college and shop activity systems simultaneously and how individuals in each group developed their knowledge differently, particularly in relation to *formal* mathematics, due to their different motivations and developing identities. This transition of the person in relation to knowledge, Beach suggests, can be considered as horizontal development, as individuals, with possibly different motivations at different times, need to be able to deconstruct and reconstruct mathematical understandings in different settings, drawing on the resources that each provides. This re-casts the problem of knowledge transfer as one of knowledge transformation and gives recognition and value to making meaning of the relation between knowledge and the situation in which it is applied. In terms of the person and their relationship with knowledge in transition, this suggests that we need to move away from only attributing value to the usual conception of developmental progress as constantly being *upwards and onwards*. In relation to preparation for work and change within work, we also need to value horizontal development that results in enriched mathematical understanding and application.

This focus on individual developmental progress, with its implicit notions of identity development, has parallels with aspects of Engeström's (2001) work that considers the *expansive learning* of a community, with its members reflecting upon the object of their activity and responding to the associated problems, systemic contradictions and the resulting personal conflicts that arise, in order to develop a new *expanded* model of their activity.

Recognising and valuing such notions of horizontal development provides a particular challenge, as it is vertical development, with its implicit upward motion through a range of settings, based on a hierarchy of knowledge, skills and understanding that become more and more generalised, abstract and distant from the specifics of human activity, that provides the dominant notion of human progress in learning in educational settings (particularly in mathematics and science). In preparation for transition from school to work, it seems important, therefore, that we give increased recognition and value to the effort required when transforming or creating new relations between knowledge and social activities that do not have conventional and implicit value in a vertical sense. This needs to be considered as progress, as suggested by Beach (1999) and Engeström (2001), amongst others. In particular, when such an effort is proving problematic for an individual, it ought not to be seen as a deficiency but rather as something to be valued and supported; indeed, it needs to be given space and recognised as an essential aspect of learning activity. In general, crossing boundaries between activity systems, whether these are vertically/hierarchically or horizontally/non-hierarchically related, has the potential for generating learning: Such boundary crossing requires the transformation of existing, or construction of new, knowledge and a shift or development in identity.

Following from the above, this suggests that boundary objects (Star, 1989) that have meaning and use in different communities have a potentially important role to play. For example, graphs as boundary objects, whilst having very distinctive, and often very different, features that make them useful in mediating meaning within each particular community (see, e.g., Roth & McGinn, 1998; Williams et al., 2001), also have common underlying structural features that are maintained across the communities in which they are used. Boundary objects should not be considered to be constituted as having the same meaning in each community because in reality, they will have significant, sometimes subtle, differences that are often problematic to boundary crossers. For example, in recent research exploring mathematical learning in college and university settings, with the implicit notion of vertical progression, it was found that the notion of *inverse function* was very differently conceived of by communities in college and in university settings (Jooganah & Williams, 2010). Indeed, although inverse function was initially considered by teacher and students as an unproblematic boundary object (the assumption in each community was that an inverse function would be understood as the same in both), this in reality was not the case. The difference in understanding of inverse function between the university academic and his new students coming directly from school/college emerged through discussion in a problem class and provoked contradictions and conflict for members of the university community. The resolution resulted in expansive learning, with members of that community coming to understand issues underpinning, and differences between, learning and mathematical understanding in university and in school/college. However, as Hoyles et al. (2010) recognised, boundary objects can provide an effective way forward. In facilitating mathematical learning in the workplace, they take the view that well-designed symbolic artefacts acting as boundary objects can assist in developing shared meanings and productive learning across communities. Hoyles et al. also argue that the reverse case of badly designed boundary objects can result in the breakdown of such shared meanings, understanding and, consequently, of learning.

In recognition of the importance that artefacts/instruments as boundary objects play, Kerosuo and Toiviainen (2011) draw attention to two types of boundaries that they identify in learning (as either horizontal or vertical development): *socio-spatial* and *instrumental–developmental* boundaries. Socio-spatial boundaries are highly visible and therefore relatively obvious; for example, school/college students may be involved in clearly defined communities focused on education (school/college) and (part-time) work. On the other hand, instrumental–developmental boundaries emphasise the developmental role that artefacts have to play in supporting boundary crossing: recognition of this seems particularly important in considering how learning within a school setting might prepare such students for future boundary crossing. In a study that investigated the adoption of a new artefact in a workplace setting, Hasu (2000) demonstrated how this presented a major challenge to a workplace community of practice, and was only accomplished through collective visualisation and reflective dialogue, with expansion of the object of activity. Such findings suggest that curriculum design must recognise the central role that artefacts play in motivating and influencing learning activity in ways that facilitate expansion of the object of activity in the learning system and consequently impact upon the collective motive and purpose of the community (Leont'ev, 1978). Attention needs to be paid to how artefacts embody mathematics and to how these require adaptation to the different contexts in which they are used.

5 Learning as doing and becoming

Having made a case for greater attention to be paid to the notion of horizontal development of knowledge and the person, I wish to focus a little more closely on the person as a learner in any setting. In relation to their learning, engagement in *doing* has long been recognised: Greek playwright Sophocles (c. 496–406 B.C.) is quoted as saying, "One must learn by doing the thing; for though you think you know it, you have no certainty, until you try". However, such active engagement is not sufficient for learning; learning requires reflection by the learner if they are to make new meaning of the knowledge, skills and understanding they engage with. When space is provided for reflection on doing/practice, as Bakker and Akkerman (2013) argue, there is the potential for taking different perspectives on that practice, leading to learning. In relation to this, Wenger (1998) suggests that learning is fundamentally experienced and social, and transforms the learner's identity, as Hahn (2011) also recognises. Black et al. (2010) explored learners' transitions in relation to mathematics from school through college to university (16–19 years) and found that identity formation was crucial in mediating their relationships with mathematics. In particular, it was found that a student's aspirations play a fundamental role in determining these relationships. Building on the work of Stetsenko and Arievitch (2004), who developed the construct of leading activity, we suggest that learners at any one time have a leading identity (Black et al., 2010) in relation to mathematics and that this reflects their hierarchy of motives. The crucial factor is how a learner values mathematics; in particular, for its *use* or for its *exchange* value. We found that some learners value mathematics for its use when they can see its immediate or potential application within the field of mathematics itself or within another area of study or work. However, because of the role of mathematics in school settings as ultimately providing a high-stake *gatekeeper* qualification, for many, its exchange value in terms of certification provides motivation for its study. Accordingly, any general mathematics curriculum aiming to prepare learners for workplace practice should seek to support identity development through a use orientation; however, this use value of mathematics should also be valued in terms of its potential exchange value.

Crucially, learning involves both *doing* and *becoming*. Learning trajectories are not outcomes of passivity: they require agency, engagement and direction. Wenger (1998) suggests that learning needs to be "designed *for*" (p. 229), seeking to provide affordances rather than constraints for identity transformation. Wenger's position supports at a meta-level the perspective argued for by Roth and McGinn (1997) in relation to the specific mathematical practice of graphing. Their analysis from a sociocultural perspective leads them to suggest that graphs as productions have three major purposes: (a) as semiotic devices, in the sense of Peirce, as signs representing aspects of reality; (b) as rhetorical devices used to communicate or elaborate their author's construction of science; and (c) as conscription devices that in their production stimulate and facilitate the activity of scientific communities. This view of graphing as practice offers contextual support to learning both as doing and as becoming. Engaging students in the practice of graphing, specified and formulated to emphasise these key features of engaging in *doing* graphing and *becoming* someone who uses and communicates with others who do likewise, may ultimately allow students to develop an insight into mathematics as an important tool in workplace practice. As Roth and McGinn recognise, this would enable novice students to become successful members in a community of graphing practice. This practice perspective may be developed to cover a wide range of other practices, such as spreadsheet building, developing spatial representations, presenting statistical information and so on. In many ways, it mirrors key aspects of both general mathematical competences (Williams, Wake, & Jervis, 1999) (discussed below) and techno-mathematical literacies (Hoyles et al., 2010) that focus on how mathematics may be an object of study but also involve activity that enables the development of models of reality. Crucially, these constructs in their different ways position learning mathematics as encompassing more than exposure to content and the practising of a range of process skills. Each pays attention to the learner developing in a transformative way new relationships between mathematics and new contexts. This suggests that new approaches to strategic curriculum design are required that focus on these important aspects of learning mathematics for application in a range of settings.

6 Principles for strategic curriculum design

Up to this point, I have attempted to distil what might be learned from an understanding of mathematics as it manifests itself in workplace practice and learning. I have introduced some important theoretical conceptions that might inform some, necessarily general, principles for curriculum design. In addition to providing strategic direction, these principles might inform the specification of curricula at the tactical design stage in ways that facilitate the competences and subjective needs of students as they become workers (and, more generally, citizens) with the capacity to develop mathematical understanding of, and facility with, new practices with which they might engage.

I propose the following principles for strategic curriculum design:

1. Curriculum specification should be formulated so that mathematics is conceptualised as more than an object of study and take account of *mathematics in practice*. In doing so, mathematical *activity* should be elaborated and supported, and learner identity development or transformation in becoming a user of mathematics should be recognised and facilitated.
2. Ways in which artefacts can be used to embody both mathematics and context, and allow communication of the reality they re-present, should be emphasised. In doing so, the ways in which the activity of the production of such artefacts supports the development of a community of practice should be recognised.
3. Horizontal mathematisation across a wide range of different contexts should be recognised and valued. Consequently, mathematical models and their relationship with the realities they represent should be emphasised and explored, with learners constructing models of their own and developing familiarity with important and recurring models.
4. Making sense of the mathematical productions (models) of others is a particularly important activity. Learners should routinely be expected to make sense of (i.e., deconstruct) and critique the mathematical activity and productions of others and build on these to provide new jointly engineered productions as part of mathematical communication.
5. The technology which “black-boxes” mathematics transforms mathematical activity by requiring sense making of symbolic productions of the technology in relation to the input and the use of these productions to communicate meaning in terms of contextual factors. The mathematics curriculum needs to attend to strategies and skills that are required when working in this way.

7 Strategic to tactical curriculum design: from foundations to structural development

I have so far provided the background case for an expansion of the school curriculum beyond usual formulations and elicited five general principles for a redesign at a strategic level. However, current curriculum formulations provide a well-defined and widely known epistemology and, most importantly, support a discourse that allows us to discuss mathematics as a knowledge domain with meaning across broad and different constituencies. The following is pragmatically situated in, and builds on, this discourse.

In general, mathematics as a domain is usually considered to be focused on, and specified in terms of, key *content* areas as knowledge sub-domains. At a meta-level, these are often identified as number, algebra, geometry, probability and statistics. However, particularly in the last 5–10 years or so, curricula that are narrowly focused around content areas have been questioned, and the ensuing debate has recognised that learners need to be better equipped to be able to *apply* mathematics in a range of contexts, so that they can become critically enquiring, problem-solving citizens of the future (Rocard et al., 2007). In many cases, this observation has led to new formulations and specifications of curricula that include and explicitly identify a range of problem-solving competencies or process skills. For example, in England, the 2007 curriculum specification (QCDA, 2007) focused on the key processes of a mathematical problem-solving cycle, involving the processes of representing, analysing, interpreting and evaluating and, at a meta-level, communicating and reflecting.

Central to arguments about how to proceed in curriculum development has been the unjustified dichotomy in the mathematics education community, and more widely, drawn between mathematics content and process skills. Sometimes, this is couched in terms of *mastering the basics* (generally relating to application of procedural techniques) or developing *skills* in problem solving and modelling. The demand for mathematical literacy (Steen, 1990) has recently seen many countries adopting more process-oriented mathematics curricula (Dorier, 2010). Terms such as *quantitative literacy*, *mathematical literacy*, *numeracy* and *functional mathematics* have been used to try to capture the essence of what might inform a new curriculum that ensures that people are better equipped to use mathematical knowledge and skills in ways that might empower them to solve problems and be able to make critical and informed choices based on quantitative information. The PISA studies, which seek to measure students' mathematical ability at age 15, "to use their knowledge and skills to meet real-life challenges, rather than merely the extent to which they have mastered a specific school curriculum" (Organisation for Economic Co-operation and Development, 2004, p. 20), have been a key driver in nations giving priority to process skills and problem solving.

It is clear that prioritising *either* technical fluency with mathematical content *or* process skills is a dangerous route to take, given that mathematical activity requires a blending of engagement with (a) mathematical content, (b) mathematical competencies and (c) context. It is in the interplay of these components, not only internally within the mathematics, but also externally within the social setting in which they are situated, that we encounter some of the key issues in the dialectic between mathematical practices in academic and workplace settings. It is important to recognise the interdependency of these components.

The analysis of college-workplace case studies by Wake and Williams (2001), drawing on our earlier experience of pre-vocational curriculum research and development, led us to propose the construct of general mathematical competences (g.m.c.) (Williams et al., 1999). These provide, at a tactical level, a potential structure for the design of curricula (and materials) that might, with appropriate adaptation, embody the strategic design principles listed above. They focus curriculum specification on attempting to define forms of task formulation that consider ways in which learners and workers might commonly bring together *content* and *competencies* when applying mathematics in different *contexts* (e.g., in pre-vocational learning in science and technical areas). For example, technicians in laboratory settings are often involved in practices that involve them in handling data graphically, almost certainly using technology, although this activity might look very different from its nearest equivalent in school/college. In addition to *handling (experimental) data graphically*, we identified six other competences:

- Costing a project (e.g., in terms of money, materials, energy use, etc.)
- Interpreting large data sets
- Using mathematical diagrams
- Using models of direct proportion
- Measuring quantities
- Using formulae

Each g.m.c. may apply in many different social settings and contexts and is, therefore, sufficiently general to be useful; each organises a substantial body of mathematical concepts and competencies and emphasises the use of mathematical models, paying attention to their underlying assumptions and validity (Williams et al., 1999). Fundamentally, the task formulation attempts to ensure that the mathematical activity leads to meaningful insights into mathematical ways of working that might pertain in different (including vocational) settings. Consequently, the brief titles of the proposed g.m.c. hide a wealth of detail that each encapsulates. For example, in the development of curriculum materials at a technical design stage, we encouraged students to develop critical enquiry skills, engaging them in mathematical comprehension of existing graphical representations, statistical diagrams and so on. Although the underpinning content is already included in current curricula formulations, it is the unfamiliar way to students (but familiar to practitioners) in which it is required to be brought together that often proves problematic. For a clear illustration of this, consider the relatively straightforward school mathematics involved in the railway signal engineer vignette provided above. In particular, our research suggests the need, at the tactical level of curriculum design, for formulations that require learners to engage with a range of skills and strategies used to deconstruct and reconstruct mathematical meaning. We have identified some strategies that were useful in such activities (Wake & Williams, 2001, 2003): for example, anchoring and bridging to common or everyday experiences; considering simple, special or very large and small cases; questioning validity and so on.

General mathematical competences were developed to reflect our observation that in the workplace, mathematical activity becomes subsidiary to the overall objective of activity. For example, in school/college, students may be concerned with making sense of information from, and ways of working in, another discipline; whereas, in the workplace, they may be concerned with production of some sort. The mathematics becomes an integrated part of the activity to such an extent that the boundaries between knowledge disciplines become blurred. Also, and importantly, although individuals require mathematical resources upon which they need to draw (e.g., how to convert quantities between different sets of units), these form only a subsidiary part of the overall activity. Not only do g.m.c. offer a useful tactical design model by which to organise and define a mathematics curriculum, but they might also meet the strategic design principles earlier identified. To add some detail to this contention, I turn now to illustrate how the general mathematical competence of *measuring quantities* might be developed according to these principles so as to provide students with mathematical activity that might better prepare them for the challenges that mathematics might pose if they were to work elsewhere, for example, in railway signal engineering.

Issues of measurement and measures underpin much workplace activity in all spheres (e.g., Bakker, Wijers, Jonker, & Akkerman, 2011; Cockcroft, 1982). Throughout all workplace settings, there is increasing measurement of workplace performance and its effectiveness. Although understanding the nature of the input data may be unproblematic (e.g., the distances used in railway signal engineering activity), understanding output measures and how they relate to input data, let alone the processing of these, often provides considerable challenge. In the context of measures, therefore, the design principles for curriculum suggest that:

1. The production of measures is recognised as a mathematical practice and provides a tactical organising structure within the curriculum. Learners should consequently have opportunities to engage in both the production/development *and* exploration of the use of measures across a range of different settings and contexts.
2. Measures as mathematical productions should be seen and appreciated as artefacts that reflect the context from which they arise and can be used to communicate aspects of the reality they represent. How the mathematical structure of the measure interconnects with the structure of this reality should be emphasised and explored. Here, boundary objects have an important role to play.

3. The practice of production of a measure as one potential outcome of mathematical modelling should be recognised and appreciated as such. Learners should have opportunities to experience measure production and measures as models arising in (many) different settings (e.g., the range of indices used in finance) and have space to reflect on common mathematical principles that are important in such practices (see below).
4. In addition to the production of measures, learners should have opportunities to deconstruct and critique the mathematical structure of measures that are productions of others. For example, in the case of average gradient, as calculated by the railway signal engineer, learners could explore an alternative model that involves working with much more detailed data.
5. Measures are often black-boxed in technology, and this needs to be recognised as a feature of the application of mathematics in many settings. Students may, for example, consider how they might be expected to programme a spreadsheet to calculate average gradients or, alternatively, explore a black-box model that is supplied to them. Their developing understanding of how processed data output relate to raw input data is crucial.

Although the workplace activity of calculating a measure of average gradient, as identified in the work of the railway signal engineer, may provide the stimulus for the production of learning materials (at a level of technical design), it is the general mathematical competence focused on the production of measures that I wish to emphasise as important in the tactical design of curriculum structure. The importance of how this curriculum structure is communicated should not be underestimated. It is essential that this communication emphasises learner activity (doing) and how this activity can form an essential aspect of induction into communities (becoming).

To illustrate issues that occur in the next step, developing further from tactical to technical design, it is worth considering the production of average gradient in just a little more detail. This particular case raises issues of weighting in the production of measures which we found to be important in a number of case studies. In the particular case of average gradient, this relied on deep and structural understanding of the two concepts of gradient and average measures of location and their interconnection. In another case study, where the weighting of data became significant, an engineer modelled how much space was filled by equipment across 13 separate areas, referred to as *nodes*, in the workplace. This activity was part of a larger problem in which the engineer was trying to determine the effect of a break in pipes carrying steam. The engineer had calculated 13 individual percentage values and went on to explain to researcher and student:

> So, what we could have done, is just added up these 13 numbers and divided it by 13 and found an average of the 13. But, when you look at it, I mean, the percentage…. If node nine is 25% full of equipment, then that's a lot of equipment compared to, say, node one[2] having 25% of equipment. So we couldn't really just add them all up and divide by 13 because it's…

Once again, in this case, a student involved was unsure why it was not correct simply to find an average of the values for each area. Here, as in the case of average gradient, contents across sub-domains of mathematics interact: percentages of areas and volumes interact with measures of location (averages). Although context and setting provide challenges in terms of understanding, they may also provide the potential for learners to develop their mathematical understanding of content working with a range of competencies. At the level of technical

[2] A scale diagram of the situation showed node 9 to have a much larger floor area than node 1.

curriculum design, it is important for curriculum developers to have insight into emerging priorities from current workplace practices that incorporate mathematics. Although I have indicated some of these at a strategic level, such as the need for the unpacking of existing mathematical productions, the case of the weighting of data in the production of measures illustrates the need to re-examine accounts of existing workplace practices in ways that might inform detailed technical curriculum design.

8 The way forward: an expanded research agenda

It is essential that future general mathematics curriculum specifications recognise the quickly changing nature and wide range of contexts in which mathematical principles are being applied. One only has to consider how every aspect of our lives in the industrialised world has been changing rapidly in this and the last century: from the way we are now interconnected by technology in our leisure time through to how we can work across geographical and temporal boundaries in ways that only a few years ago were impossible, if not inconceivable. Technology is key and provides the potential for unbounded creativity in the form of quantitative and representational productions that are possible. In workplaces, this can be seen in the output of production and the generation of services that are increasingly monitored, controlled and communicated. It is possible that school mathematics as a distinctive genre could remain isolated from the rapid technology-driven advances that are taking place in many aspects of our lives. However, it is necessary to recognise the potential that the use of technology affords for providing innovative mathematical productions and to recognise the difficulty that many workers experience in being able to use mathematics to make sense of these with ease and facility.

I have argued in this article that mathematics curricula need to reposition mathematics as a discipline that builds connections; indeed, that it should be at the nexus of our interaction as individuals and communities with each other and with a range of situations and contexts. It is important, therefore, to design curricula in ways that ensure that mathematics is valued by learners as they attempt to make sense of, and with, mathematics in ways that facilitate their being able to engage in practice (doing) and developing their identity (becoming). However, such an approach to the formulation of mathematics curricula signals that we need an expanded research agenda that allows a better understanding of the learning of mathematics in its juxtaposition with context. It is important not only to form a better understanding of an individual's relationship with mathematics and context, but also how this is socially constituted in ways that allow for shared understanding and communication. Attempting to inform curriculum design for general education in education systems that appear increasingly inextricably situated in performance-driven contexts provides considerable challenge, especially where curricula focused on knowledge recall and technical facility with rules and procedures tend to prevail. In attempting to meet the challenges likely to be faced by such an agenda for change, it is necessary to draw on an enhanced understanding of how individuals can react to, and be supported by, pedagogies that are informed by, and designed to capture the essence of, new conceptualisations of mathematical competences. These competences need to encompass an enhanced range of activities that provide learners with opportunities, when working with others, to make mathematical productions that represent a range of realities, to make sense of such mathematical productions as devised by others and to communicate effectively. Research that explores activity in these new spaces of educational design in order to support mathematical activity of this type is required. The collective papers of this special issue provide an indication of starting points for such an endeavour, but the challenge is considerable and should not be underestimated.

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