Algebra Standard for Grades 9–12

	Expectations
Instructional programs from prekindergarten through grade 12 should enable all students to—	**In grades 9–12 all students should—**
Understand patterns, relations, and functions	• generalize patterns using explicitly defined and recursively defined functions; • understand relations and functions and select, convert flexibly among, and use various representations for them; • analyze functions of one variable by investigating rates of change, intercepts, zeros, asymptotes, and local and global behavior; • understand and perform transformations such as arithmetically combining, composing, and inverting commonly used functions, using technology to perform such operations on more-complicated symbolic expressions; • understand and compare the properties of classes of functions, including exponential, polynomial, rational, logarithmic, and periodic functions; • interpret representations of functions of two variables
Represent and analyze mathematical situations and structures using algebraic symbols	• understand the meaning of equivalent forms of expressions, equations, inequalities, and relations; • write equivalent forms of equations, inequalities, and systems of equations and solve them with fluency—mentally or with paper and pencil in simple cases and using technology in all cases; • use symbolic algebra to represent and explain mathematical relationships; • use a variety of symbolic representations, including recursive and parametric equations, for functions and relations; • judge the meaning, utility, and reasonableness of the results of symbol manipulations, including those carried out by technology.
Use mathematical models to represent and understand quantitative relationships	• identify essential quantitative relationships in a situation and determine the class or classes of functions that might model the relationships; • use symbolic expressions, including iterative and recursive forms, to represent relationships arising from various contexts;

	• draw reasonable conclusions about a situation being modeled.
Analyze change in various contexts	• approximate and interpret rates of change from graphical and numerical data.

In the vision of school mathematics in these Standards, middle-grades students will learn that patterns can be represented and analyzed mathematically. By the ninth grade, they will have represented linear functions with tables, graphs, verbal rules, and symbolic rules and worked with and interpreted these representations. They will have explored some nonlinear relationships as well.

In high school, students should have opportunities to build on these earlier experiences, both deepening their understanding of relations and functions and expanding their repertoire of familiar functions. Students should use technological tools to represent and study the behavior of polynomial, exponential, rational, and periodic functions, among others. They will learn to combine functions, express them in equivalent forms, compose them, and find inverses where possible. As they do so, they will come to understand the concept of a class of functions and learn to recognize the characteristics of various classes.

High school algebra also should provide students with insights into mathematical abstraction and structure. In grades 9–12, students should develop an understanding of the algebraic properties that govern the manipulation of symbols in expressions, equations, and inequalities. They should become fluent in performing such manipulations by appropriate means—mentally, by hand, or by machine—to solve equations and inequalities, to generate equivalent forms of expressions or functions, or to prove general results.

The expanded class of functions available to high school students for mathematical modeling should provide them with a versatile and powerful means for analyzing and describing their world. With utilities for symbol manipulation, graphing, and curve fitting and with programmable software and spreadsheets to represent iterative processes, students can model and analyze a wide range of phenomena. These mathematical tools can help students develop a deeper understanding of real-world phenomena. At the same time, working in real-world contexts may help students make sense of the underlying mathematical concepts and may foster an appreciation of those concepts.

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Understand patterns, relations, and functions

High school students' algebra experience should enable them to create and use tabular, symbolic, graphical, and verbal representations and to analyze and understand patterns, relations, and functions with more sophistication than in the middle grades. In helping high school students learn about the characteristics of particular classes of functions, teachers may find it helpful to compare and contrast situations that are modeled by functions from various classes. For example, the functions that model the essential features of the situations in figure 7.4 are quite different from one another. Students should be able to express them using tables, graphs, and symbols.

Situation 1: In February 2000 the cost of sending a letter by first-class mail was 33¢ for the first ounce and an additional 22¢ for each additional ounce or portion thereof through 13 ounces.

Number of ounces	1	2	3	4	5	...	*P*
Cost in cents	33	33 + 22	33 + 2(22)	33 + 3(22)	33 + 4(22)	...	33 + (*P* – 1)(22)

Situation 2: During 1999 the population of the world hit 6 billion. The expected average rate of growth is predicted to be 2 percent a year.

Situation 3: A table of data gives the number of minutes of daylight in Chicago, Illinois, every other day from 1 January 2000 through 30 December 2000.

551, 553, 555, 557, 559, 562, 565, 568, 571, 575, 579, 582, 586, 591, 595, 599, 604, 609, 614, 619, 624, 629, 634, 639, 644, 650, 655, 661, 666, 672, 677, 683, 689, 694, 700, 706, 711, 717, 723, 728, 734, 740, 745, 751, 757, 762, 768, 773, 779, 785, 790, 796, 801, 806, 812, 817, 822, 827, 832, 837, 842, 847, 852, 856, 861, 865, 870, 874, 878, 881, 885, 889, 892, 895, 896, 901, 903, 905, 907, 909, 911, 912, 913, 914, 914, 914, 914, 914, 914, 913, 912, 911 909, 907, 905, 903, 901, 898, 895, 892, 889, 885, 882, 878, 874, 870, 866, 861, 857, 852, 848, 843, 838, 833, 826, 823, 818, 813, 807, 802, 797, 791, 786, 781, 775, 770, 764, 758, 753, 747, 742, 736, 731, 725, 719, 714, 708, 703, 697, 691, 686, 680, 675, 669, 664, 658, 653, 648, 642, 637, 632, 627, 622, 617, 612, 607, 603, 598, 594, 590, 585, 581, 578, 574, 571, 567, 564, 561, 559, 557, 554, 553, 551, 550, 549, 548, 547, 547, 547, 548, 548, 549, 550

Fig. **7.4.** Three situations that can be modeled by functions of different classes

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For the first situation, students might begin by generating a table of values. If *C* is the cost in cents of mailing a letter and *P* is the weight of the letter in ounces, then the function $C = 33 + (P - 1)(22)$ describes *C* as a function of *P* for positive integer values of *P* up through 13. » Students should understand that this situation has some linear qualities. For real-number values of *P,* the points on the graph of $C = 33 + (P - 1)(22)$ lie on a line, and the rate of change is constant at 22 cents per ounce. However, the actual cost of postage and the linear function agree only at positive integer values of *P.* Students must realize that the graph of postal cost as a function of weight is a step function, as seen in figure 7.5.

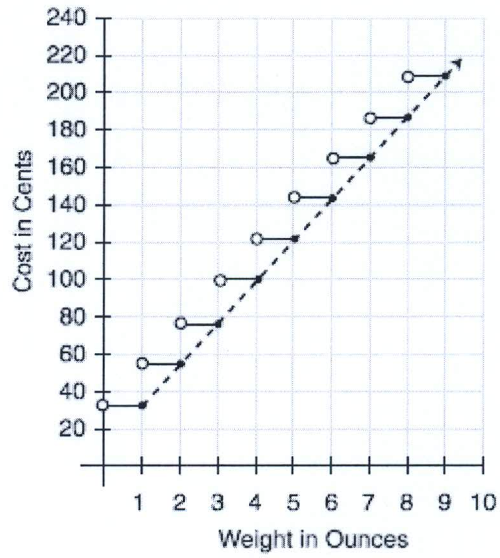

Fig. **7.5.** A comparison of step and linear functions

For the second situation described in figure 7.4, teachers could encourage students to find a general expression for the function and note how its form differs from the step function that describes the postal cost. Some students might generate an iterative or recursive definition for the function, using the population of a given year (NOW) to determine the population of the next year (NEXT):

NEXT = (1.02) • NOW, start at 6 billion

(See the discussion of NOW-NEXT equations in the "Representation" section of chapter 6.) Moreover, students should be able to recognize that this situation can be represented explicitly by the exponential function $f(n) = 6(1.02)^n$, where $f(n)$ is the population in billions and n is the number of years since 1999. A discussion of whether this formula is likely to be a good model forever would help students see the limitations of mathematical models.

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For the third situation, students could begin by graphing the given data. It will help them to know that everywhere on earth except at the equator, the period of sunlight during the day increases for six months of the year and decreases for the other six. From the graph, they should be able to see that the daily increase in daylight is nonconstant over the first half of the year and that the decrease in the second half of the year also is nonconstant. Students could be asked to find a function that models the data well. The teacher could tell them that the length of » daylight can indeed be modeled by a function of the form $T(t) = T_{ave} + T_A(\theta) \sin(\omega t + \varphi)$, where t is the time, T_{ave} = average daylight time = 12 hours; $T_A(\theta)$ = amplitude, depending on latitude θ (changes sign at the equator); ω = frequency = $2\pi/(12 \text{ months})$, and φ = phase (depending on choice of the initial time, t_0). Students will see such formulas in their physics courses and need to understand that formulas express models of physical phenomena. It is also important to note that the parameters in physical equations have units.

After exploring and modeling each of the three situations individually, students could be asked to compare the situations. For example, they might be asked to find characteristics that are common to two or more of the functions. Some students might note that over the intervals given, the first function is nondecreasing, the second is strictly increasing, and the third both increases and decreases. Students need to be sensitive to the facts that functions that are increasing over some intervals don't necessarily stay increasing and that increasing functions may have very different rates of increase, as these three examples illustrate.

Students could also be asked to consider the advantages and disadvantages of the different ways the three functions were represented. The teacher should help students realize that depending on what one wants to know, different representations of these functions can be more or less useful. For instance, a table may be the most convenient way to initially represent the postage function in the first example. The same may be so for the third example if the goal is to determine quickly how much sunlight there will be on a given day. Despite the convenience of being able to "read" a value directly, however, the table may obscure the periodicity of the phenomenon. The periodicity becomes apparent when the function is represented graphically or symbolically. Similarly, although students may first create tables when presented with the second situation, graphical and symbolic representations of the exponential function may help students develop a better understanding of the nature of exponential growth.

High school students should have substantial experience in exploring the properties of different classes of functions. For instance, they should learn that the function $f(x) = x^2 - 2x - 3$ is quadratic, that its graph is a parabola, and that the graph opens "up" because the leading coefficient is positive. They should also learn that some quadratic equations do not have real roots and that this characteristic corresponds to the fact that their graphs do not cross the x-axis. And they should be able to identify the complex roots of such quadratics.

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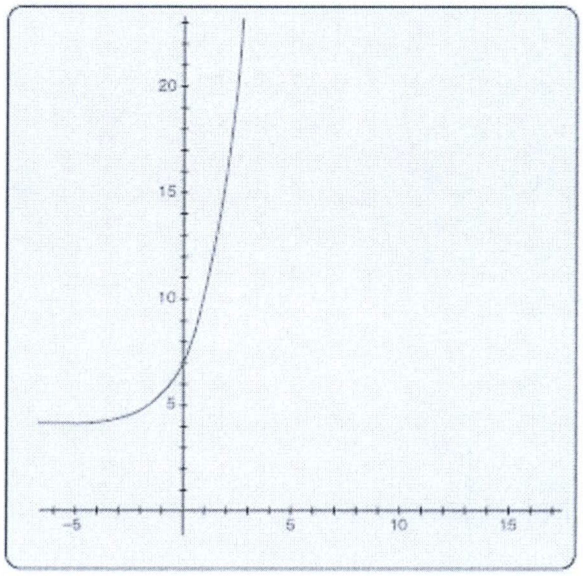
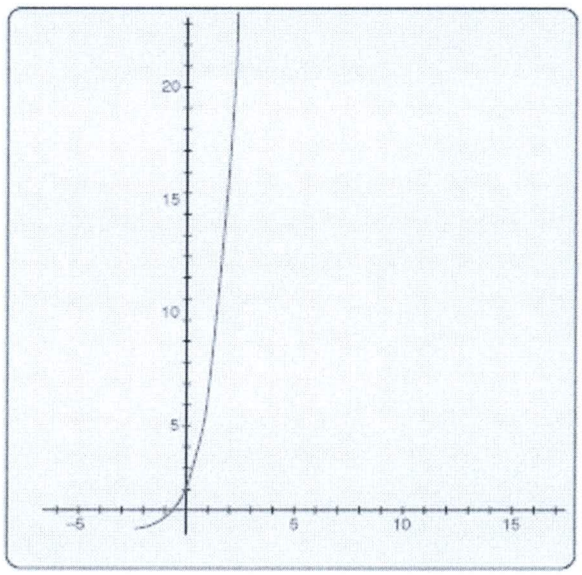
In addition, students should learn to recognize how the values of parameters shape the graphs of functions in a class. With access to computer algebra systems (CAS)—software on either a computer or calculator that carries out manipulations of symbolic expressions or equations, can compute or approximate values of functions or solutions to equations, and can graph functions and relations—students can easily explore the effects of changes in parameter as a means of better understanding classes of functions. For example, explorations with functions of the form $y = ax^2 + bx + c$ lead to some interesting results. The consequences of changes in the parameters a and c on the graphs of functions are relatively easy to observe. Changes in b are not as obvious: changing b results in a translation of the parabola along a » nonvertical line. Moreover, a trace of the vertices of the parabolas formed as b is varied forms a parabola itself. Exploring functions of the form $f(x) = a(x - h)^2 + b(x - h) + c$ and seeing how their graphs change as the value of h is changed also provides a basis for understanding transformations and coordinate changes.

As high school students study several classes of functions and become familiar with the properties of each, they should begin to see that classifying functions as linear, quadratic, or exponential makes sense because the functions in each of these classes share important attributes. Many of these attributes are global characteristics of the functions. Consider, for example, the graphs of the three exponential functions of the form $f(x) = a \cdot b^x + c$, with $a > 0$ and $b > 1$, given in figure 7.6.



$g(x) = 3 \cdot 2^x + 4$



$h(x) = 2 \cdot 3^x - 1$

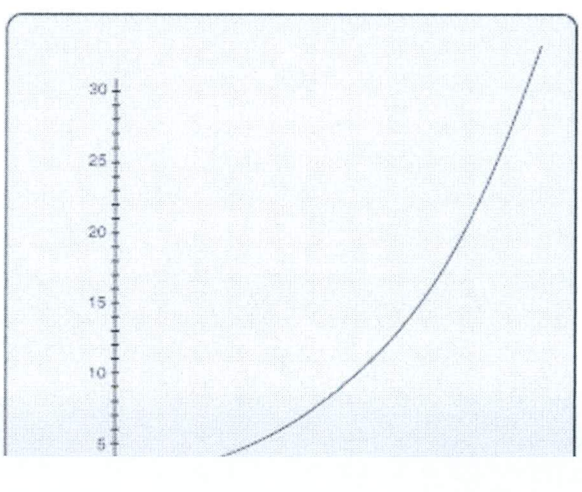


Fig. **7.6.** Graphs of exponential functions of the form $f(x) = a \cdot b^x + c$

To help students notice and describe characteristics of these three functions, teachers might ask, "What happens to each of these functions for large positive values of x? For large negative values of x? Where do they cross the y-axis?" One student might note that the values of each function increase rapidly for large positive values of x. Another student could point out that the y-intercept of each graph appears to be $a + c$. Teachers should then encourage students to explore what happens in cases where $a < 0$ or $0 < b < 1$. Students should find that changing the sign of a will reflect the graph over a horizontal line, whereas changing b to $1/b$ will reflect the graph over the y-axis. The graphs will retain the same shape. This type of exploration should help students see that all functions of the form $f(x) = a \cdot b^x + c$ share certain properties. Through analytic and exploratory work, students can learn the properties of this and other classes of functions.

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Represent and analyze mathematical situations and structures using algebraic symbols

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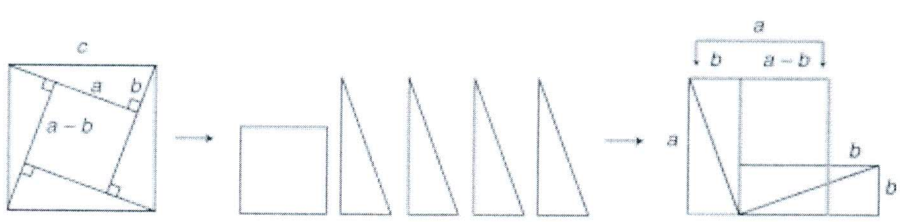
Fluency with algebraic symbolism helps students represent and solve problems in many areas of the curriculum. For example, proving that the square of any odd integer is 1 more than a multiple of 8 (see the related discussion in the "Number" section of this chapter) can involve representing odd numbers and operating on that representation algebraically. Likewise, the equations in figure 7.7 suggest an algebraic justification of » a visual argument for the Pythagorean theorem. And many geometric conjectures—for example, that the medians of a triangle intersect at a point—can be proved by representing the situation using coordinates and manipulating the resulting symbolic forms (see the "Geometry" section of this chapter). Straightforward algebraic arguments can be used to show how the mean and standard deviation of a data set change if sample measurements are converted from square meters to square feet (see the "Reasoning and Proof" section of this chapter).



$$c^2 = (a-b)^2 + 4\left(\frac{1}{2}ab\right) = a^2 + b^2$$

Fig. **7.7.** An algebraic explanation of a visual proof of the Pythagorean theorem

Students should be able to operate fluently on algebraic expressions, combining them and reexpressing them in alternative forms. These skills underlie the ability to find exact solutions for equations, a goal that has always been at the heart of the algebra curriculum. Even solving equations such as

$$(x+1)^2+(x-2)+7=3(x-3)^2+4(x+5)+1$$

requires some degree of fluency. Finding and understanding the meaning of the solution of an equation such as

$$e^{4x}=4e^{2x}+3$$

calls for seeing that the equation can be written as a quadratic equation by making the substitution $u = e^{2x}$. (Such an equation deserves careful attention because one of the roots of the quadratic is negative.) Whether they solve equations mentally, by hand, or using CAS, students should develop an ease with symbols that enables them to represent situations symbolically, to select appropriate methods of solution, and to judge whether the results are plausible.

Being able to operate with algebraic symbols is also important because the ability to rewrite algebraic expressions enables students to reexpress functions in ways that reveal different types of information about them. For example, given the quadratic function $f(x) = x^2 - 2x - 3$, some of whose graphical properties were discussed earlier, students should be able to reexpress it as $f(x) = (x - 1)^2 - 4$, a form from which they can easily identify the vertex of the parabola. And they should also be able to express the function in the form $f(x) = (x - 3)(x + 1)$ and thus identify its roots as $x = 3$ and $x = -1$.

p. 301 The following example of how symbol-manipulation skills and the ability to interpret graphs could work in concert is a hypothetical composite of exploratory classroom activities, inspired by Waits and Demana (1998): »

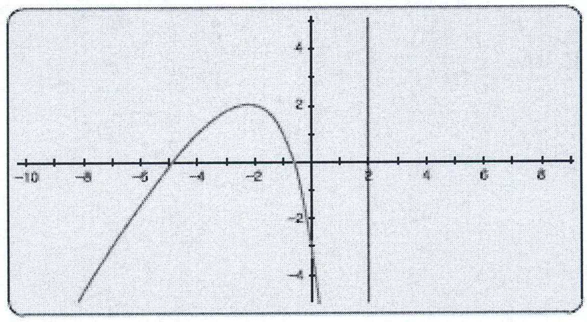
A teacher asks her students to analyze the function

$$f(x)=\frac{2x^2+11x+6}{x-2}$$

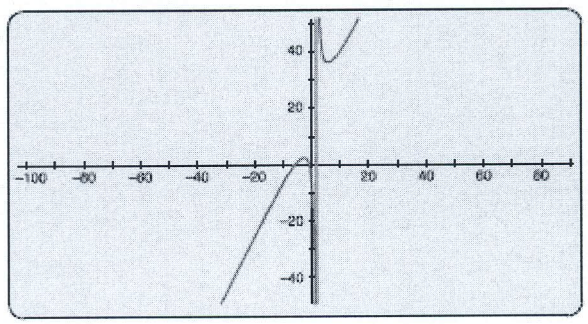
and make as many observations about it as they can. Some students begin by trying to graph the function, plotting points by hand. Some students use a CAS and others perform long division by hand, producing the equivalent form

$$f(x)=2x+15+\frac{36}{(x-2)}.$$

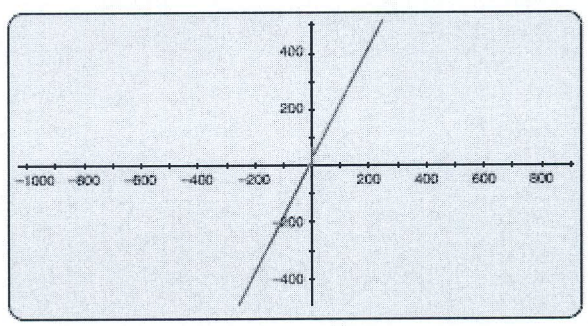
Some graph the original function or the equivalent form on a computer or on graphing calculators; the zoom feature enables them to see various views of the graph, as seen in figure 7.8.



(a)



(b)



(c)

Fig. **7.8.** Different views of the function

$$f(x) = \frac{2x^2 + 11x + 6}{x - 2}$$

It is hard to interpret some of the graphs near $x = 2$, a matter the class returns to later. Focusing on a graph where the zoom-out feature has been used a number of times (see fig. 7.8c), some students observe, "The graph looks like a straight line." The teacher asks the class to decide whether it is a line and, if so, what the equation of the line might be. To investigate the question, the teacher suggests that they find several values of *f(x)* for large positive and negative values of *x* and use curve-fitting software to find the equation of the line passing through those points. Different groups choose different *x*-values and, as a result, obtain slightly different values for the slope and the *y*-intercept. However, when the class discusses their findings, they discover that the lines that fit those points all seemed "close" to the line $y = 2x + 15$. Some students point out that this function is part of the result they obtained after performing the long division.

The class concludes that the line $y = 2x + 15$ is a good *approximation* to *f(x)* for large *x*-values but that it is not a perfect fit. This conclusion leads to the question of how the students might combine the graphs of $g(x) = 2x + 15$ and $h(x) = 36/(x - 2)$ to deduce the shape of the graph of *f(x)*. Hand-drawn and computer plots help students explore how the graph of each function "contributes to" the graph of the sum. Examining the behavior of

$$h(x) = \frac{36}{x-2}$$

leads to a discussion of what "really" happens near $x = 2$, of why the function appears to be linear for large values of *x*, and of the need to develop a sense of how algebraic and graphical representations of functions are related, even when graphing programs or calculators are available.

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Students in grades 9–12 should develop understandings of algebraic concepts and skill in manipulating symbols that will serve them in situations that require both. Success in the example shown in figure 7.9, for example, requires more than symbol manipulation. There are several ways to approach this problem, each of which requires understanding algebraic concepts *and* facility with algebraic symbols. For example, to complete the first row of the table, students need only know how to evaluate *f(x)* and *g(x)* for a given value of *x*. However, to complete the second row, students must know what it means to compose functions, including the role of the "inner" and "outer" function and the numbers » on which they act in a composition. They also must understand how to read the symbols *fg(x))* and *g(f(x))*. Students might reason, using an intuitive understanding of the inverse of a function, that because $g(x) = 4$, *x* must be either 1 or –3. They can then determine that *x* cannot be 1, because *g(f(1))* is not 81.

If $f(x) = x^2 - 1$ and $g(x) = (x + 1)^2$, complete the table below.

x	$f(x)$	$g(x)$	$f(g(x))$	$g(f(x))$
2			80	16
		4		81

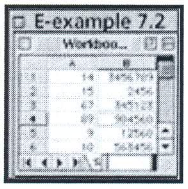
Fig. **7.9.** A composition-of-functions problem (Adapted from Tucker [1995])

Top

Use mathematical models to represent and understand quantitative relationships

Modeling involves identifying and selecting relevant features of a real-world situation, representing those features symbolically, analyzing and reasoning about the model and the characteristics of the situation, and considering the accuracy and limitations of the model. In the program proposed here, middle-grades students will have used linear functions to model a range of phenomena and explore some nonlinear phenomena. High school students should study modeling in greater depth, generating or using data and exploring which kinds of functions best fit or model those data.

Teachers may find that having students generate data helps generate interest in creating mathematical models. For example, students could conduct an experiment to study the relationship between the time it takes a skateboard to roll down a ramp of fixed length and the height of the ramp (Zbiek and Heid 1990). Teams of students might set ramps at different heights and repeatedly roll skateboards down the ramps and measure the time. Once students have gathered and plotted the data, they can analyze the physical features of the situation to create appropriate mathematical models. Their knowledge of the characteristics of various classes of functions should help them select potential models. In this situation, as the height of the ramp is increased, less time is needed, suggesting that the function is decreasing. Students can discuss the suitability of linear, quadratic, exponential, and rational functions by arguing from their data or from the physics of the situation. Curve-fitting software allows students to generate possible models, which they can examine for suitability on the basis of the data and the situation.



Medicine: Applying Graphs, Tables and Equations

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In making choices about what kinds of situations students will model, teachers should include examples in which models can be expressed in iterative, or recursive, form. Consider the following example, adapted from National Research Council (1998, p. 80), of the elimination of a medicine from the circulatory system.

> A student strained her knee in an intramural volleyball game, and her doctor prescribed an anti-inflammatory drug to reduce the swelling. She is to » take two 220-milligram tablets every 8 hours for 10 days. If her kidneys filtered 60% of this drug from her body every 8 hours, how much of the drug was in her system after 10 days? How much of the drug would have been in her system if she had continued to take the drug for a year?

Teachers might ask students to conjecture about how much of the drug would be in the volleyball player's system after 10 days. They might also ask about whether the drug keeps accumulating noticeably in the athlete's system. Students will tend to predict that it does, and they can be asked to examine the accumulation in their analysis.

Students might begin by calculating a few values of the amount of the drug in the player's system and looking for a pattern. They can proceed to model the situation directly, representing it informally as

NEXT = 0.4(NOW) + 440, start at 440

or more formally as

$$a_1 = 440 \text{ and } a_{n+1} = 0.4a_n + 440 \text{ for } 1 \leq n \leq 31,$$

where n represents the dose number (dose 31 would be taken at 240 hours, or 10 days) and a_n represents the amount of the drug in the system just *after* the nth dose. By looking at calculator or spreadsheet computations like those in figure 7.10, students should be able to see that the amount of the drug in the bloodstream reaches an after-dosage "equilibrium" value of about 733 1/3 milligrams. Students should learn to express the relationship in one of the iterative forms given above. Then the mathematics in this example can be pursued in various ways. At the most elementary level, the students can simply verify the equilibrium value by showing that 0.4(733 1/3) + 440 = 733 1/3 milligrams. They can be asked to predict what would happen if the initial dose of the anti-inflammatory drug were different, to run the simulation, and to explain the result they obtain.

	A	B
1	440	
2	616	
3	686.4	
4	714.56	
5	725.824	
6	730.3296	
7	732.13184	
8	732.852736	
9	733.1410944	
10	733.2564376	
11	733.3025751	
12	733.32103	
13	733.328412	
14	733.3313648	
15	733.3325459	
16	733.3330184	
17	733.3332073	
18	733.3332829	
19	733.3333132	
20	733.3333253	
21	733.3333301	
22	733.333332	
23	733.3333328	
24	733.3333331	
25		

Fig. **7.10.** A spreadsheet computation of the "drug dosage" problem

This investigation opens the door to explorations of finite sequences and series and to the informal consideration of limits. (For example, spreadsheet printouts for "large *n*" for various dosages strongly suggest that the sequence $\{a_n\}$ of after-dosage levels converges.) Expanding the first few terms reveals that this is a finite geometric series:

$$a_1 = 440 = 440(1)$$

$$a_2 = 440 + 0.4(440) = 440(1 + 0.4)$$

$$a_3 = 440 + 0.4(440) + (0.4)^2(440) = 440\left(1 + 0.4 + (0.4)^2\right)$$

$$a_4 = 440 + 0.4(440) + (0.4)^2(440) + (0.4)^3(440)$$
$$= 440\left(1 + 0.4 + (0.4)^2 + (0.4)^3\right)$$

Students might find it interesting to pursue the behavior of this series.

To investigate other aspects of the modeling situation, students could also be asked to address questions like the following:

- If the athlete stops taking the drug after 10 days, how long does it take for her system to eliminate most of the drug?

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- How could you determine a dosage that would result in a targeted after-dosage equilibrium level, such as 500 milligrams? »

Students should also be made aware that problems such as this describe only one part of a treatment regimen and that doctors would be alert to the possibility and implications of various complicating factors.

In grades 9–12, students should encounter a wide variety of situations that can be modeled recursively, such as interest-rate problems or situations involving the logistic equation for growth. The study of recursive patterns should build during the years from ninth through twelfth grade. Students often see trends in data by noticing change in the form of differences or ratios (How much more or less? How many times more or less?). Recursively defined functions offer students a natural way to express these relationships and to see how some functions can be defined recursively as well as explicitly.

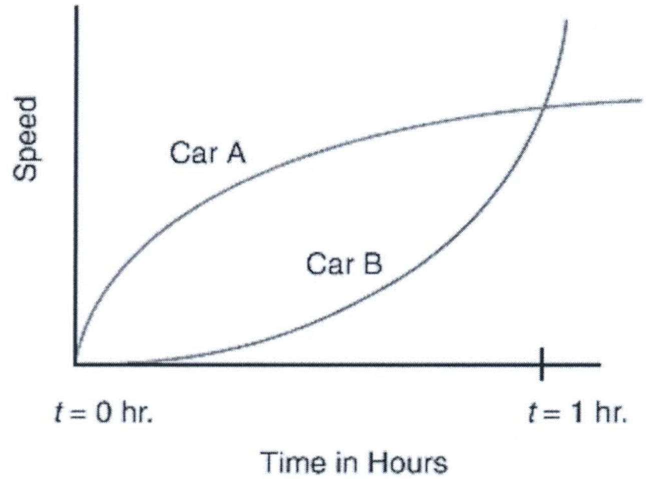
Top

Analyze change in various contexts

Increasingly, discussions of change are found in the popular press and news reports. Students should be able to interpret statements such as "the rate of inflation is decreasing." The study of change in grades 9–12 is intended to give students a deeper understanding of the ways in which changes in quantities can be represented mathematically and of the concept of rate of change.

The "Algebra" section of this chapter began with examples of three different real-world contexts in which very different kinds of change occurred. One situation was modeled by a step function, one by an exponential function, and one by a periodic function. Each of these functions changes in different ways over the interval given. As discussed earlier, students should recognize that the step function is nonlinear but that it has some linear qualities. To many students, the kind of change described in the second situation sounds linear: "Each year the population changes by 2 percent." However, the change is 2 percent of the previous year's population; as the population grows, the increase grows as well. Students should come to realize that functions of this type grow *very* rapidly. In the third example, students can see that not only is the function periodic but because it is, its rate of change is periodic as well.

Chapter 6 gives an example in which middle-grades students are asked to compare the costs of two different pricing schemes for telephone calls: a flat rate of $0.45 a minute versus a rate of $0.50 a minute for the first 60 minutes and $0.10 a minute for each minute thereafter. In examples of this type, the dependent variable typically changes (over some interval) a fixed amount for each unit change in the independent variable. In high school, students should analyze situations in which quantities change in much more complex ways and in which the relationships between quantities and their rates of change are more subtle. Consider, for example, the situation (adapted from Carlson [1998, p. 147]) in figure 7.11.



The given graph represents velocity vs. time for two cars. Assume that the cars start from the same position and are traveling in the same direction.

(*a*) State the relationship between the position of car A and that of car B at $t = 1$ hr. Explain.

(*b*) State the relationship between the velocity of car A and that of car B at $t = 1$ hr. Explain.

(*c*) State the relationship between the acceleration of car A and that of car B at $t = 1$ hr. Explain.

(*d*) How are the positions of the two cars related during the time interval between $t = 0.75$ hr. and $t = 1$ hr.? (That is, is one car pulling away from the other?) Explain.

Fig. **7.11.** A problem requiring a sophisticated understanding of change

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Working problems of this type builds on the understandings of change developed in the middle grades and lays groundwork for the study of calculus. Because students tend to confuse velocity with position, teachers should help them think carefully about which variables are represented in the diagram and about how they change. First, for example, students must realize that the variable on the vertical axis is velocity, rather than position. To answer part *a* of the question, they need to reason that because the velocity of car A is greater than that of car B at every point in the interval $0 < t < 1$, car A has necessarily traveled a » greater distance than car B. They can read the answer to part *b* directly off the graph: at $t = 1$ hour, both cars are traveling at the same velocity. Answering part *c* calls for at least an intuitive understanding of instantaneous rate of change. Acceleration is the rate of change of velocity. At $t = 1$ hour, the velocity of car B is increasing more rapidly than that of car A, so car B is accelerating more rapidly than car A at $t = 1$ hour. Part *d* is particularly counterintuitive for students (Carlson 1998). Since car B is accelerating more rapidly than car A near $t = 1$ hour, students tend to think that car B is "catching up" with car A, and it is, although it is still far behind. Some will interpret the intersection of the graphs to mean that the cars meet. Teachers need to help students focus on the relative velocities of the two cars. Questions such as "Which car is moving faster over the interval from $t = 0.75$ hour to $t = 1$ hour?" can help students realize that car A is not only ahead of car B but moving faster and hence pulling away from car B. Car B starts catching up with car A only after $t = 1$ hour.