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What makes a counterexample exemplary?

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Abstract In this paper we describe two episodes of instructional interaction, in which examples are used in order to help students face their misconceptions. We introduce the notions of pivotal example and bridging example and highlight their role in creating and resolving a cognitive conflict. We suggest that the convincing power of counterexamples depends on the extent to which they are in accord with individuals' example spaces.

Keywords Cognitive conflict · Conceptual change · Pivotal example · Bridging example · Example space · Proof scheme

1 Introduction

Examples are relevant to a variety of engagements with mathematics: from learning and striving to understand, to teaching, designing curriculum and inventing. Though attention to examples is a recurrent theme in mathematics education research, there appears to be a renewed interest in the use of examples in learning mathematics; both in examples generated upon invitation by learners (Watson and Mason 2005) and examples used by teachers in an instructional setting (Zaslavsky and Lavie 2005). This trend is further evidenced in a recent research forum on exemplification at the 30th Conference of the International Group for Psychology in Mathematics Education (Bills, Dreyfus, Mason, Tsamir, Watson & Zaslavsky 2006).

Acknowledging the centrality of examples, in this paper we attend to their convincing power in confronting learners' incorrect mathematical derivations. As suggested by the title, we echo John Mason's question of "What makes an example exemplary" (Mason 2002, 2006)? He suggests that it "is seeing it as a particular case or instance of a more general class of objects; being aware of what can be varied and still it belongs to the class, and within what range of values it can be varied" (2006, p. 62). We extend these considerations to counterexamples by introducing the notion of a *pivotal example*. We build on the ideas of

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cognitive conflict and conceptual change and discuss the role of counterexamples with respect to these theoretical constructs.

2 Conceptual change and cognitive conflict

The term *conceptual change* is used to characterize "the kind of learning required when the new information to be learned comes in conflict with the learners' prior knowledge usually acquired on the basis of everyday experiences" (Vosniadou and Verschaffel 2004, p. 445). Conducting research with prospective teachers, rather than with young learners, our perspective on the notion of conceptual change highlights the importance of a conflict – a cognitive conflict – between information and experience. However, the information does not have to be 'new,' but 'newly realized' or 'newly attended to,' and the experience may come from prior learning opportunities rather than from everyday engagement.

Studies of Piaget outlined the importance of the state of balance for cognitive growth, a balance that is achieved through accommodating/assimilating towards equilibration by meeting the challenges of disequilibration. *Cognitive conflict* is an analogue of disequilibration, referring to a pedagogical setting and a learner's cognitive development. A cognitive conflict is invoked when a learner is faced with contradiction or inconsistency in his or her ideas. It is important to mention that learners may possess conflicting ideas, and co-existence of these ideas may not be acknowledged and thus will not create a dissonance. However, inconsistency of ideas presents a *potential conflict*, it becomes a *cognitive conflict* only when explicitly invoked, usually in an instructional situation. With general acceptance of a variety of constructivists' perspectives, using cognitive conflict techniques has become a desirable pedagogical strategy to remedy misconceptions (Ernest 1996). This approach allows students to trouble their own thinking, and it is through this conflict that they develop their own meanings, or at least seek to rectify the conflict. Implementing a cognitive conflict approach has been reported in studies on a variety of topics, such as division (Tirosh and Graeber 1990), sampling and chance in statistics (Watson 2002), limits (Tall 1977) and infinity (Tsamir and Tirosh 1999).

When errors arise from some misconception, it is appropriate to expose the conflict and help the learner to achieve a resolution (Bell 1993). However, while there is some understanding how a cognitive conflict can be exposed, once a potential conflict is recognized, there is little knowledge on how to help students in resolving the conflict. In this article we introduce the notion of "pivotal-bridging example" as a possible means towards conflict resolution.

3 The role of examples

The central role of examples in teaching and learning mathematics has long been acknowledged. It is impossible to consider teaching and learning mathematics without consideration of specific examples. Examples are said to be an important component of expert knowledge. They are used to verify statements, to illustrate algorithms and procedures, and to provide specific cases that fit the requirements of the definition under discussion (Rissland 1991). Mason and Pimm (1984) suggest that when a teacher presents an example, he or she sees its generality and relates to what the example represents. However, a learner may notice only particular features of a specific example, paying attention to the example itself and not to what it stands for. Therefore, Mason and Pimm advocate the use of so called 'generic examples,' examples that represent the general case and attempt to ignore the specifics of the example itself.

An additional role of examples is in changing one's mind or ways of operation. Examples that involve large numbers may help learners seek better approaches for a particular solution. Counterexamples may help learners readjust their perceptions or beliefs about the nature of mathematical objects. Further, the role of counterexamples has been acknowledged and discussed in creating a cognitive conflict (Klymchuk 2001; Peled and Zaslavsky 1997; Zaslavsky and Ron 1998). However, counterexamples may not be sufficient in a conflict resolution, which is an ultimate goal of instruction. As teachers, we are to seek strategic examples that will contribute not only to invoking a cognitive conflict but also to resolving it. To highlight the issue, we introduce two theoretical constructs: pivotal example and pivotal-bridging example.

4 Pivotal and bridging examples

As mentioned above, counterexamples may serve teachers in implementing a powerful instructional strategy, that of creating a cognitive conflict. However, researchers in mathematics education are well aware that learners may possess contradicting ideas without facing or acknowledging a conflict. As such, a counterexample, when presented to a learner, may not create a cognitive conflict; it may be simply dismissed or treated as exception. A *pivotal example* may be needed.

An example is *pivotal* for a learner if it creates a turning point in the learner's cognitive perception or in his or her problem solving approaches; such examples may introduce a conflict or may resolve it. In other words, pivotal examples are examples that help learners in achieving what is referred to as "conceptual change" (e.g. Tirosh and Tsamir 2004; Vosniadou and Verschaffel 2004). When a pivotal example assists in conflict resolution we refer to it as a *pivotal-bridging example*, or simply *bridging example*, that is, an example that serves as a bridge from learner's initial (naïve, incorrect or incomplete conceptions) towards appropriate mathematical conceptions.

We note here that while a counterexample is a mathematical concept, pivotal example is a pedagogical concept. As such, what constitutes an appropriate counterexample – an example that satisfies the conditions of a conjecture, but violates the conclusion – can be determined universally and a-priori. However, whether a given example serves as a pivotal or bridging example for a learner cannot be determined, only anticipated, before implementation, and can be fully recognized only after implementation in an instructional setting.

Our notion of pivotal-bridging example is connected to Mason and Pimm's notion of a "generic example," but it shifts the attention from the features of an example to the role an example plays for a learner. From a pedagogical perspective, "generic example" – example that represents the general case and attempts to ignore the specifics of the example itself – is useful as a tool to move from specific to general when proving conjectures or describing relationships. Similarly, our notion of bridging example is a tool to move from personal (naïve or incomplete) conceptions to conventional (mathematically appropriate) conceptions.

In what follows we exemplify the notions of pivotal example and pivotal-bridging example in two episodes of instructional interactions. Our data was not collected with the specific purpose of testing the notion of a pivotal example; rather, the new theoretical constructs emerged through the data analysis. The episodes feature two preservice elementary school teachers: Selina and Tanya. Interaction with Selina took place in a clinical interview setting as a part of an ongoing research on learning elementary number theory. Interaction with Tanya occurred as part of classroom discussion in a mathematics methods course for prospective elementary school teachers. In each case we start with introducing the student and the setting, then present the interaction and conclude with analysis of the interaction focusing on the role of examples.

The two episodes – which can be seen as two separate case studies – are connected by what Mason refers to as "methodology of noticing." Following Mason (2006), our "method of enquiry is to identify phenomena [we] wish to study, and to seek examples within [our] own experience" (p. 43). However, while Mason applies this method when offering mathematical tasks, we offer examples of interaction between the expert and the learner. We describe and analyse two episodes, with the expectation that the readers may recognise similar experiences in their own practice. In doing so our goal is to "highlight or even awaken sensitivities and awarenesses for them" (Mason 2006 p. 43), as well as to provide a theoretical tool to analyse the experience. Though we do not suggest that these episodes are generalisable, we suggest that increased awareness will have an impact on future practice for both teachers and researchers, in choosing examples and in analysing exemplification, respectively.

5 Episode 1: Selina and prime numbers

5.1 Introducing Selina

Selina is a prospective elementary school teacher in her late-twenties enrolled in a course "Principles of Mathematics for Teachers," which is a core course in the teacher certification program at the elementary level. During the course, and prior to the interview, the students studied a chapter on elementary number theory that included divisibility and divisibility rules, prime and composite numbers, factors and multiples, prime decomposition and the fundamental theorem of arithmetic.

Selina's achievement in the course was above average, she had a positive attitude and willingness to engage with the subject matter, though she acknowledged not being exposed to any mathematical content since her high school graduation. As part of ongoing research on prospective teachers' mathematical content knowledge, we invited students to participate in a clinical interview. Specifically, our focus was on how prospective teachers develop or change their understanding of number structure when presented with particular mathematical tasks. Among the volunteers for the clinical interview Selina appeared as being able to articulate clearly her thinking about the problems and her solution approaches. In analyzing one of the tasks that Selina was asked to entertain during the interview, we noticed a cognitive conflict that Selina faced and resolved. We have chosen this particular task as it introduces minor novelty in a familiar setting; we have chosen this particular episode as it allows us to follow Selina's conflict in reconciling robust concepts with a novel realization.

5.2 The task

The task presented to Selina asked to simplify $\frac{13\times17}{19\times23}$. In choosing tasks to engage students in a clinical interview setting we focus on what Zazkis and Hazzan (1998) identified as "familiar with a twist." The chosen task is familiar in a way that simplifying numerical expressions in general, and reducing fractions in particular, does not present novelty in the assignment. However, the twist is in the non-standard form the fraction is presented – both numerator and denominator being products of two primes. We were interested to see whether the students would notice that the numbers involved are prime and, in fact, there is no possibility to reduce the fraction. Therefore, considering prime decomposition, the fraction is already presented in its simplest form, or alternatively, the form achieved by multiplying the numbers in the numerator and the denominator, $\frac{221}{437}$, is the reduced simplified form.

5.3 Is 437 prime? Identifying potential conflict

Selina's work on the task starts by multiplying out the numbers and then searching for common factors in the numerator and denominator.

> Selina: So 13×17 is 221, yeah okay, and then 19×23, um, (pause), I'm allowed to use the calculator.
>
> Interviewer: Absolutely...
>
> Selina: Okay, it's 437. Okay, so we have 221/437, neither of them are divisible by 2, neither of them are divisible by, well this, no it's not divisible by 3 because this equals 5, um, and this equals 14, so I'll try...
>
> Interviewer: What do you mean equals 5, equals 14...
>
> Selina: Well I'm checking for divisibility by 3, so I'm adding these three numbers, to see if it's divisible by 3, um, so it's not..., 221 divisible by 4, don't think so, I'll check, no it's a decimal. Um, well see I think that, I think that 437 is a prime number, I have to see what 37 divided by 2 is, 18.5. I'm going to test it (pause), 437 divided by, let's just say 7...
>
> Interviewer: You said you are going to test it, what are you testing?
>
> Selina: Oh I'm testing whether this number is prime or not, 437, whether it's prime... So, um, just trial and error, I'm just going to try dividing it..., because it's an odd number, I'm going to try dividing it by um certain prime numbers that we know, like 13, that equals a decimal, divided by 17, that equals a decimal, um, divided by (pause) hmm, I don't know, (pause) I would say that's it, I don't know what else, how else to find, to find that.
>
> Interviewer: So you started to check whether 437 was prime...
>
> Selina: Um hm...
>
> Interviewer: Is it?
>
> Selina: I don't know, I'm getting lazy again. So it doesn't work for 13, it doesn't work for 17, let's try for 11, no, let's try, I tried it for 7 already, it's definitely not 5, it's definitely not 3, it's definitely not 2, um, (pause) well we know it divides, 19 divides 437 23 times because that's in the original equation...
>
> Interviewer: So is it prime?
>
> Selina: Yes it is, because it's two prime numbers, of course it is, because two prime numbers multiplied by each other are prime, (pause).

Selina immediately recognizes that 221 and 437 are not divisible by 2 and then proceeds to check divisibility by 3. It is interesting to note that after assuring that 437 is not divisible by 3 (using divisibility rules) and by 4 (using a calculator), she then comes up with a conjecture that 437 is prime. Nevertheless, she keeps checking this number's divisibility by 7, 13 and 17.

Having tried 19, she confirms that "that's in the original equation", meaning that divisibility of 437 by 19 can be concluded from its being calculated as 19×23. At this moment she acknowledges for the first time her awareness that 19 and 23 are prime, but draws a conclusion that "two prime numbers multiplied by each other are prime." Such a derivation was acknowledged in prior research (Zazkis and Liljedahl 2004) and was described as intuitive "tendency towards closure," that is, a tendency to view the result of an operation between two elements of a set as an element of the same set. However, Selina's explicit acknowledgement of this intuitive belief invited intervention. In what follows we present the interviewer's attempt to invoke a cognitive conflict by focusing on a strategic example.

5.4 Is 15 prime? Pivotal example and invoking cognitive conflict

> Interviewer: Is 15 a prime number?
>
> Selina: No.
>
> Interviewer: But it's two prime numbers multiplied by each other, 3 and 5...
>
> Selina: But (pause) something about, 2 and 3 are tricky because they're, they're, I found that in my brain in looking up prime numbers 2 and 3 and 5, 15 isn't a prime number, yet it is the product of two prime numbers, but (pause) these numbers have a common factor of 1 and only 1, so (pause) I can't, I can't make into words what it is that I want to say...
>
> Interviewer: Can you make me pictures (laugh)...
>
> Selina: Okay. 2 and 3, they're prime numbers, 19, 23, 13 and 17 are all prime numbers, um, $2 \times 3 = 6$, but 6 is not a prime number, so the theory that any prime, (pause) so the theory that it's a closed set, it's not a closed set. So prime numbers under multiplication aren't necessarily a closed set, aren't a closed set, because there's the disproof of it. So...
>
> Interviewer: In what way 19 multiplied by 23 is different from 2 multiplied by 3, and what I gave you before, 3 multiplied by 5? You claim 6 is not prime, you claim 15 is not prime,...
>
> Selina: Um hm...
>
> Interviewer: And still you suggested 437 is prime. So my question is, what is different?
>
> Selina: Well you can, well that you can't divide it by 2, 3 or 5...
>
> Interviewer: Yeah, but you can divide by 19...
>
> Selina: (pause) I kind of see 2, 3 and 5 as building blocks to all other, uh other numbers, that's kind of the way I see it. And I find that once you can eliminate those as options, then you're dealing with, then you're dealing with prime numbers that, I (pause), I don't know how to say it, I don't know how to say it, it's frustrating (laugh).

The interviewer's strategic example is *pivotal* for Selina: it introduces a cognitive conflict and challenges her initial ideas. When faced with a conflict, she acknowledges her difficulty with frustration. Selina recognizes that 15 and 6 are not prime numbers, even though they are products of two primes. So this disproves her initial suggestion that a product of primes is prime, and she phrases it appropriately utilizing recently acquired terminology, "prime numbers under multiplication aren't a closed set." Following the request to describe in what way 2, 3 or 5 are different from 19, she alludes to her initial belief that the number 437 is prime because "you can't divide it by 2, 3 or 5." It was described in prior research that, in an attempt to check whether a given number is prime, students check the number's divisibility only by "small primes" (Zazkis and Liljedahl 2004). The revealing point in Selina's description is that this way of thinking is explicitly acknowledged rather than derived from students' actions and that the list of "small primes" is limited to 2, 3 and 5. At this time the interviewer employed a technique described by Ginsburg (1997) as 'establishing the strength of belief,' by introducing an additional strategic example. The intention in using this technique is to determine whether a specific response given by the interviewee is consistent with his or her responses to similar tasks and in such whether it represents a developed strategy or conviction.

5.5 How about 77? Pivotal-bridging example and conflict resolution

Interviewer: Oh, I don't want to frustrate you, but it is very interesting what you're saying. How about 77?

Selina: 77 isn't prime because it's divisible by 11.

Interviewer: Oh I see, so is 11 one of your building blocks?

Selina: 11 is, (pause) I mean in this case 11 and 7 also factor out, also act the same way, oh no because, no I don't think 7 and 11 are building blocks that I'm talking about. I find 2, 3 and 5 are.

Interviewer: But is 77 prime?

Selina: No.

Interviewer: How about 221?

Selina: 221 I don't think is prime.

Interviewer: Why do you think these are not prime?

Selina: (pause) Well it can't (pause), no okay they're not prime, because they're not, they have more factors than just 1, right, so 437 can't be prime because its factor is 19 and 23. So prime meaning that it can only be multiplied the number by itself, right, so these aren't prime. The 437 isn't prime, (pause) so it changes everything.

Interviewer: What does it mean, it changes everything?

Selina: Well it changes what I first said, because I said that two numbers multiplied by each other, two prime numbers multiplied by each other would equal a prime number, but I was wrong in saying that, totally wrong.

We suggest that the interviewer's choice of 77 served as a *bridging example* for Selina in the resolution of the conflict. From a perspective of elementary number theory, the numbers mentioned in this interview excerpt – 6, 15, 77, 221 and 437 – are similar in their structure of prime decomposition. All these numbers are products of two prime numbers. So in what sense is 77 different from 6 and 15 on the one hand and different from 437 on the other hand? We suggest that on one hand this number is "small enough" that it is similar to 6 and 15 (but unlike 437) because its factors are immediately recognizable. On the other hand it is not composed of 2, 3 or 5, the numbers that Selina referred to as "building blocks." As such, this number serves as a *pivotal-bridging example* for Selina as it "changes everything," that is, it changes her initial suggestion of a product of primes being a prime. We note that her earlier observation, that "prime numbers under multiplication aren't *necessarily* a closed set," is based on considering a counterexample. She draws this conclusion, most likely realizing that a product of primes *may be* not a prime number. It is the example of 77 that "changes everything." It makes the initial assumption "totally wrong" for Selina and guides her towards a correct conclusion. Based on our interpretation of the language used by prospective elementary school teachers, we suggest that "totally wrong" means "wrong in all cases," rather than wrong because of the existence of several counterexamples. At this stage Selina is referred to the initial problem of simplifying the given expression.

5.6 Can you simplify? Potential conflict

> Interviewer: So let's go back to our problem of simplifying this number, you perform multiplication in the numerator and the denominator to get this thing, 221/437. The question is, can you simplify it?
>
> Selina: No, because these four numbers are prime, you can't simplify because you would have to be, this number [pointing to 221]would have to be divisible by either of these two numbers [pointing to 19 and 23], and we'll test that. So 221 divided by 19 equals, so that is a decimal, so that doesn't work. 221 divided by 23 is also a decimal, so that doesn't work.

Here, on one hand, Selina claims that the fraction cannot be simplified, but on the other hand she finds it necessary to confirm with the help of a calculator that 221 is not divisible by either 19 or 23. She was probed further for her initial claim.

> Interviewer: Is it possible to know that it will not work, what was just checked, without working with the calculator and without doing long division?
>
> Selina: (pause) I'm sure there is a pattern somewhere, but I don't, I don't quite see it. [pause] Well you know what, see I'll see 3×7 is 21 here and I see this 9×3 is 7, which 9×3 really is 27, so that might explain why this 7 is here, um, I'd go maybe look and that might be the direction that I would start looking in, these digits here, the 3, the 7, the 9 and the 3, I would look at that, just try and figure that out. But that's as far as I could go with it.

Is Selina applying, implicitly, the fundamental theory of arithmetic, in her claim that "because these four numbers are prime, you can't simplify?" The idea that prime decomposition is unique, as formalized by the fundamental theorem of arithmetic, was discussed and exemplified in class prior to the interviews. However, this idea is usually not accepted intuitively by students, especially when numbers larger than 10 are involved (Zazkis and Campbell 1996, Zazkis and Liljedahl 2004). Being invited to explain why the product of 13 and 17 is not divisible by 19, Selina regresses to considering the last digits in multiplication, being "sure there is a pattern somewhere." We recognize a potential conflict in Selina's awareness that the numbers are prime, yet her need to perform division to confirm lack of divisibility. However, this conflict was neither invoked nor resolved during the interview.

It was the interviewer's choice, reading Selina's declining interest in the task, not to press the issue any further at the given moment. However, the question remains: How could it have become a cognitive conflict? More specifically, are there strategic examples that can help Selina recognize the inefficiency of her approach of "checking divisibility?" As we have shown, different examples, while seen as similar by an expert, may have different convincing power for a student. We further illustrate this difference in analyzing interaction with Tanya in another instructional context.

6 Episode 2: Tanya and comparing fractions

6.1 Introducing Tanya

Tanya is a prospective elementary school teacher in her early thirties enrolled in a course "Designs for Learning: Elementary Mathematics." This is a "methods" course in which students examine topics from the elementary school curriculum with the double purpose of (1) exposure to different pedagogical approaches and (2) opportunity to strengthen and enrich their own mathematics. In the terms of Shulman (1986, 1987), acquiring curricular knowledge in this course serves as a vehicle to both develop pedagogical content knowledge and strengthen subject matter knowledge. Tanya had an extensive experience in a tutoring centre before entering our teacher education program. She was often interested in sharing with her classmates her experience with young learners, to which she referred as "tricks of the trade." Overall, she had a positive attitude towards mathematics and teaching mathematics and was respected by her classmates as a source of ideas; however, she appeared at times confined by her experience in not being sufficiently open to new ideas and strategies presented in the course. Given the nature of the course, the topic of fractions was addressed in considerable detail because the related concepts are known to be problematic to both young learners and preservice elementary school teachers. The episode described below was noticed as it represents a rather common incorrect strategy, which is rarely verbalized explicitly by a prospective teacher.

6.2 "Different strategy": identifying potential conflict

One particular classroom session focused on a variety of ways to compare fractions, such as "bench mark," "complement to a whole" and "common numerator" strategy. The goal in the presented comparison tasks was to avoid the "common denominator" strategy whenever possible, as this strategy was initially preferred by students and likely the only familiar one from their prior schooling. Towards the end of this session Tanya approached the instructor and introduced a "different strategy."

> Tanya: There is another strategy that you didn't mention, that has always worked for me.
>
> Instructor: OK, please show me.
>
> Tanya: You simply take away the top from the bottom and see what is larger. Where the number is larger, the fraction is smaller, like 2/7 and 3/7, 5 is greater than 4, so this fraction (pointing to 2/7) is smaller.
>
> Instructor: Hmm, interesting …
>
> Tanya: And the examples you showed work like that.
>
> Instructor: Would you explain why this works?
>
> Tanya: I'm not sure how to explain this, it just makes sense.

It is evident from the excerpt that Tanya is not interested in seeking explanation as to why her strategy "works." She claims it just "makes sense," and if something "works" as well as "makes sense," presumably, no explanation is needed. Furthermore, several examples discussed at that particular classroom session confirmed her strategy.

Rather than acknowledging the strategy that Tanya introduced as wrong, the instructor sought counterexamples that would demonstrate a discrepancy. In other words, having recognized the potential conflict, the instructor attempted to help Tanya face the conflict. We describe these attempts in the next section.

6.3 Different denominators? Attempts to invoke conflict

Instructor: And how about different denominators?

Tanya: Oh-yeah, it will work, it always did.

Instructor: So how about 1/2 and 2/4? Using your method we would conclude that one of these fractions is larger than another.

Tanya: But they never give you fractions that are the same to compare. So the method works when they are not the same.

Instructor: And how about 5/6 and 6/7? We have just shown how to think of them and compare without finding a common denominator. How could you apply your method in this case?

Tanya: You can't if the difference is the same. But if it is not the same, it works [pause], I think it works, it always worked for me, in school, I mean. Like 4/9 and 5/7. You said, use 1/2 as a bench mark. I just looked at 5 here and 2 here [pointing at 4/9 and 5/7] and where you get 2 the fraction is larger.

We note that, when facing a counterexample, Tanya's immediate tendency is to amend her strategy, rather than to abandon it. In the above excerpt, when presented with disconfirming evidence of 1/2 and 2/4, Tanya reduces the scope of applicability of her method by claiming "the method works when they [i.e., fractions] are not the same." Presented with further disconfirming evidence, she further reduces the scope of applicability, claiming that her strategy cannot be used "if the difference is the same." However, she immediately introduces another confirming example of 4/9 and 5/7.

6.4 Considering 9/10 and 91/100: pivotal example

Instructor: And how about something like 9/10 and 91/100?

Tanya: [pause]. So are you saying that with ridiculously large number of pieces this doesn't work?

Instructor: I'm just asking questions…

Tanya's reaction in this instance can be seen as recognition of the fact that her method is not applicable for the suggested example. It also can be seen as yet another attempt of reducing the scope of applicability. What is implicit in her words is the belief that the strategy "works" with "reasonable" numbers and not with "ridiculously large" ones. The disappointment in Tanya's voice at this point was evident, but hard to convey in writing. It may not have been the last example alone that served as a pivotal example and made Tanya reconsider her strategy, but the fact that it followed several other examples that she tried to dismiss. At this point the instructor turned to the class with the request to examine the strategy presented by Tanya. This resulted in excess of disconfirming evidence, some of which was generated by Tanya herself. Examples that included comparing 2/3 with 5/7, or 3/4 with 8/11 appeared to the students as more convincing than the initially suggested fractions of 9/10 and 91/100. Likely, these examples fall into the category of "normal" numbers rather than "ridiculously large" ones. However, rather than relative size of numbers, how can pivotal examples be characterized?

7 Discussion

In this paper we introduced the notions of *pivotal example* and *pivotal-bridging example* and illustrated the role they play in invoking or resolving a cognitive conflict. Following a comparative summary of the two episodes, we discuss the essential features of pivotal examples using the constructs of *proof schemes* (Harel and Sowder 1998) and *example spaces* (Watson and Mason 2005).

7.1 Pivotal examples and proof schemes

Selina's work on the presented task exhibited several popular misconceptions described in prior research (Zazkis and Liljedahl 2004). Her initial suggestion that 437 could be a prime number was based on a belief that any composite number should be divisible by 2, 4 or 5. She further claimed 437 was prime because it was a product of two primes. Her desire to check divisibility of 437 by 13 and 17 was based on an implicit search for an "alternative" prime decomposition. The interviewer's suggestion that 15 is also a product of two primes served as a *pivotal example* for Selina. Even though it invoked a conflict, the similarity in the structure of 15 and 437 was rejected, pointing to 2, 3 and 5 as "building blocks," which made them, in Selina's perception, different from other prime numbers. The strategic example of 77 helped Selina to reconsider her position. As such, it served as a *pivotal-bridging example*: bridging between 15 and 437 helped Selina in reconciling her initial naïve ideas with the conventional mathematics.

Tanya's strategy of comparing fractions, by looking at the difference between numerator and denominator, appeared robust when she dismissed counterexamples presented by the instructor. Rather than acknowledging the disconfirming evidence, she limited the scope of applicability of her method, by suggesting it was not applicable when fractions were equal or when the respective denominator/numerator differences were the same. The example of 9/10 vs. 91/100 did not fall into these categories and likely served for Tanya as a *pivotal example*, the first example that presented a conflict, making her doubt her strategy. Additional examples generated in the following classroom interaction suggested to Tanya that the strategy should not be just put in doubt but dismissed.

Our excerpts describe two different students that attended two different classes and interacted with an expert in two different settings, one in a clinical interview and one in a classroom. Furthermore, they proceed towards conflict resolution at a different pace, requiring a different number of examples to be considered. However, what is common in both interactions is that the initial examples chosen by the interviewer/instructor were in a way rejected by the student who preferred to consider them as exceptions rather than counterexamples.

It is evident in both interactions that the instructor/interviewer's proof scheme (Harel and Sowder 1998) is different from that of the student. "Proof scheme" signifies what is accepted as a convincing argument by an individual, and as Harel and Sowder demonstrated, could be quite different from a mathematical notion of what constitutes a proof. While existence of a counterexample provides a definite mathematical proof that a conjecture in question is false, this may not fit within a proof scheme of an individual.

The immediate attempt to confront the student with a counterexample shows that rejecting a conjecture or a claim with a counterexample is consistent with the expert's proof scheme. However, this proof scheme is not naturally possessed by learners. This explains why different counterexamples do not have the same convincing power. Different counterexamples, while serving the same mathematical purpose of rejecting a conjecture, may not be equally effective in serving a pedagogical purpose of helping a learner recognize the faulty conjecture. Notwithstanding the importance of guiding students towards the conventional proof schemes, instructors' awareness of different convincing power of different counterexamples highlights the need for instructional choices of strategic examples that may serve as pivotal examples in addressing learners' misconceptions. In what follows we suggest possible features of such examples.

7.2 Pivotal examples and example spaces

In attending to the features of examples that serve as pivotal for learners we consider the notion of Watson and Mason (2005) of *example spaces*, which are collections of examples that fulfill a specific function. Watson and Mason suggest that example spaces are influenced by an individual's experience and memory, as well as by specific requirements of tasks. Among several kinds of example spaces they distinguish between (p. 76):

1. *Situated (local), personal (individual) example spaces*, triggered by a task, cues and environment as well as by recent experience;
2. *Personal potential example space*, from which a local space is drawn, consisting of person's past experience (even though not explicitly remembered or recalled), and may not be structured in ways which afford easy access;
3. *Conventional example space,* as generally understood by mathematicians and as displayed in textbooks, into which the teacher hopes to induct his or her students.

We suggest that "conventional example spaces" – as presented in textbooks or in common instructional practices – are not representative of what is understood by experts and may be limiting the development of learners' personal example spaces. That is to say, the instructional examples often focus on "small" fractions or numbers decomposed into "small" primes, and these choices limit learners in the intended generalization. A testimony to these limiting choices is a follow up activity conducted in Tanya's class in which the students were asked to examine a page of exercises for comparing fractions in a locally used textbook. Surprisingly, about 80% of the exercises served as confirming examples for Tanya's strategy. This definitely deserves a word of caution for both teachers and textbook writers.

We suggest that in order to serve as a pivotal example, an example should fit within, but push the boundaries of personal potential example spaces. That is, this should be an example that a learner accepts as "exemplary," but usually falls outside of personal/situated (i.e., immediately available and easy accessible) example space. Further along these lines, examples that "fail" to serve as pivotal are examples that belong to conventional example spaces as understood by experts, but fall outside, at least temporarily, of personal potential example space of a learner.

Connecting the notion of a pivotal and bridging example to example spaces illustrates that this notion is learner-dependent and situation dependent, that is, a strategic example that is pivotal for one learner in a given situation may not be helpful to another learner or in another situation. As Mason noted, "exemplariness resides not in the example, but in how the example is perceived" (Mason 2006, p. 62). Further, in some cases a 'critical mass' of examples may be necessary to serve as a pivot or a bridge.

As stated earlier, our goal was to increase awareness and sensitivity to the fact that examples that are similar-mathematically can be different-pedagogically. Our explicit contribution is in providing a tool – pivotal example – to address this pedagogical disparity. This is a theoretical extension pertaining to the study of examples and counterexamples in mathematical teaching and learning as well as of means for creating and resolving a cognitive conflict.

Can teachers' awareness of potential conflict guide their choices of strategic examples? To what degree are the concepts of pivotal and pivotal-bridging example useful in invoking and resolving a cognitive conflict? Are learners' personal example spaces affected following resolution of a cognitive conflict? Future research will address these questions. However, as many cognitive conflicts are not pre-planned, seeking examples from personal experiences becomes an appropriate method of inquiry for researchers and teachers alike.

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