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Struggles With Developing the Concept of Angle: Comparing Sixth-Grade Students' Discourse to the History of the Angle Concept

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This article highlights the similarities between sixth-grade students' developing notions of angle and mathematicians' struggles to define this complex concept. Students learning from a reformed curriculum had the opportunity to voice their ideas concerning the definition of angle. These conversations were compared with past discussions and debates concerning the meaning of angle. The data were classified into three major categories: (a) What exactly is being measured when referring to the size of angles? (b) Can angles contain curves?, and (c) Difficulties with conceiving of 0°, 180°, and 360° angles. The data suggest that using multiple representations such as those examples from history would seem to be more comprehensive and a better preparation as students study the concept of angle.

Some concepts can be defined in many ways, whereas others are very straightforward. If a mathematics teacher attempts to present the meaning of acute angle to a classroom of sixth graders, the instruction would be fairly simple because there are few arguments as to the meaning of acute in the context of a geometry class; it is an angle with a measure less than 90°. However, other concepts such as the concept of *angle* take on many different meanings as they are multifaceted in nature. *Angle* has been defined in many different ways over the course of history, with the definitions varying significantly in their emphases. For example, as will be presented later in this article, some definitions focus more on the rays, whereas others are defined in terms of the rotation about a point.

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This article presents a snapshot of sixth-grade students forming their own conceptions of *angle* and compares their ideas with historical definitions of angle. The data from a dissertation study of students' conceptions of angle were reanalyzed by viewing student comments solely from a historical perspective, making comparisons whenever the students' comments were similar to historical discussions about the concept of angle.

The original dissertation study attempted to "provide an explanation and description of the learning that takes place within the environment of reflective practice of reformed teaching techniques using innovative materials" (Keiser, 1997, p. 16). Tall and Vinner's ideas (Tall, 1992; Tall & Vinner, 1981; Vinner, 1983, 1990) underlying *concept image*, *concept definition*, *compartmentalization*, and *generic organizers* had been used as the theoretical framework to analyze the data to address the primary research focus which was to provide a rich description of the understandings of *angle concepts* that sixth-grade students developed during investigations of geometric concepts in the *Shapes and Designs* unit; a unit from the Connected Mathematics Project curriculum (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1996). However, while analyzing the data, I observed similarities between students' descriptions of angle, and definitions or descriptions that had been recorded since the time of Euclid. Clements and Ellerton (1996) suggested that linking histories of a concept in mathematics to the actual instruction is a "suitable framework for achieving a more unified and more systematic approach to mathematics education research." I decided that it would be worthwhile to reanalyze the data using the history of the angle concept as the theoretical framework.

The historical comparisons made in this study are used as a means to highlight the complexity of the concept of angle, and to argue that students struggle with that complexity in their own development of the concept. If the definition of *angle* has changed over the centuries, with each definition emphasizing a different facet of the concept, then to discover that sixth-grade children still wrestle with many of these same issues is a valuable contribution to our current pedagogical content knowledge.

BACKGROUND

Haeckel's biological "law of recapitulation" from 1874 has been used as a basis for educational research that attempts to compare the cultural and historical constructions of human knowledge with the constructions that take place in our own minds as we gain in our understanding of a given topic (Barbin, 1996; Gallardo, 1994; Moreno & Waldegg, 1991; Shulman, 1995). These researchers believed that ontogeny recapitulates phylogeny, where ontogenesis is the development of an idea in an individual's mind, and phylogenesis is the development of the idea throughout history. Using this paradigm, historical studies of the issues surrounding the meaning of a certain concept have been used to point to the areas where individual students might have questions or areas of difficulty as they develop their own concept image (Tall & Vinner, 1981[1]) for that concept. Sinclair (1991), a developmental psychologist, commented,

> The difficulty of studying learning—and teaching—lies…in the fact that it demands the study of the processes by which children come to know in a short time basic principles in mathematics, but also in other scientific disciplines) that took humanity thousands of years to construct. (p. 19)

However, other researchers counter that the acquisition of a concept is completely dependent on the child's social environment. Otte (1994, p. 309) said, "The development of knowledge does not take place within the framework of natural evolution but within the frameworks of socio-cultural development" and "Knowledge is necessarily social knowledge."

What further complicates the study of students' learning of certain concepts is the fact that, depending on their use in mathematics or in other sciences, some notions continue to change in their emphases and, therefore, their definitions shift in meaning over time as well. *Angle* is one such concept that has been defined differently over the centuries and, depending on the mathematical situation at hand, can still today take on different meanings.

The data from this study came from student conversations that took place in a very social learning environment. In these classrooms, students were encouraged to negotiate meanings, to discuss their viewpoints, and to argue if they disagreed with the other members of the class. Therefore, the students were busy socially constructing their own meanings for the concept of angle. The goal of this study is to point to the historical comparisons that take place when students are given the opportunity to discuss *angle* without direct instruction from the teacher.

THE STUDY

Two sixth-grade classrooms in Michigan were observed for 5 weeks while a basic informal geometry unit from the Connected Mathematics Project (CMP), *Shapes and Designs* (Lappan et al., 1996), was being taught. The unit is the first introduction to geometry in the sixth- through eighth-grade series, and it explores two-dimensional concepts involving polygons. Students work in small groups to investigate such concepts as the Triangle Inequality, how to determine the sum of the

[1]"We shall use the term *concept image* to describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes… The *concept definition* [is] a form of words used to specify that concept." (Tall & Vinner, 1981, p. 152)

interior angles of polygons, which polygons can tessellate, and how to use benchmark angles to help estimate angle size. The two female teachers who taught these classes had been involved as pilot teachers for the draft units of CMP and were thoroughly familiar with CMP's philosophy of a student-centered curriculum. Both of the teachers (referred to as T1 and T2 in the text) were adept at involving the students and in allowing them to voice their questions and comments. A definition of *angle* was never formally presented, and students' classroom comments were recorded as they attempted to define the concept. At first, the word *angle* did not arise naturally in students' conversations, but as the curriculum materials and teaching began to introduce angle concepts, it soon became evident that collectively the students' concept images included a wide range of understandings (see Keiser, 2000).

The multifaceted nature of the angle concept is well-documented. Most classifications fit generally into one of the following three main areas: A measure of the turning of a ray about a point from one position to another (dynamic), the union of two rays with a common endpoint, and the region contained between the two rays (both static conceptions). It is because of these many interpretations that the writers of the CMP curriculum chose not to initially define *angle*. Instead, three different representations—angle as turn, angle as wedge, and angle as the intersection of lines—were integrated into the activities.

> In order to be able to have a picture of children's developing notions of a concept as complex as angle, starting with a definition and telling them a set of words is clearly not going to let you know what sense they are making of it. (CMP's *Shapes and Designs* writer, G. Lappan, personal communication, April, 1996)

ANALYSIS

After reading about the history of the angle concept and learning of the problems mathematicians had in defining *angle* too narrowly, I then returned to the student data and determined areas where students' struggles matched those from my review of the literature. After a constant comparison analysis (Glaser & Strauss, 1967), it was further noted that the similarities could be categorized into three major topics:

1. What Exactly Is Being Measured When Referring to the Size of Angles?
2. Can Angles Contain Curves?
3. Difficulties with Conceiving of 0°, 180°, and 360° Angles.

Each following section first provides a historical perspective before sharing situations faced by the students.

WHAT EXACTLY IS BEING MEASURED WHEN REFERRING TO THE SIZE OF ANGLES?

The Historical Perspective

Angle is a multifaceted concept. Defining *angle* therefore becomes a difficult process because all definitions put limitations on the concept by focusing more heavily on one facet more than any of the others. This has been the case with the early definitions of *angle*.

The early Greeks attempted to classify everything in life into one of ten Aristotelian categories.[2] Those who defined *angle* tended to weight their definition more favorably into one of these three categories—a relation, a quality, or a quantity. Though many mathematicians and philosophers put forward convincing arguments in their writings to categorize an angle in only one of these three categories, Proclus considered angle in terms of all three views. Proclus wrote,

> So the angle surely needs the underlying quantity implied in its size, it needs the quality by which it has something like a special shape and character of existence, and it needs also the relation of the lines that bound it or of the planes that enclose it. The angle is something that results from all of these, and is not just one of them. (p. 100)

By 1893, Schotten had analyzed the common conceptions of angle and found that most people adhered to one of the following three views:

1. The angle is the difference of direction between two straight lines.
2. The angle is the quantity or amount (or the measure) of the rotation necessary to bring one of its sides from its own position to that of the other side without its moving out of the plane containing both.
3. The angle is the portion of a plane included between two straight lines in the plane that meet in a point (or two rays issuing from the point; Euclid, 1956, p. 179).

In all of these different perspectives, confusion can arise when trying to identify what exactly is being measured when measuring an angle. For example, Carpus of Antioch chose to define an angle as

> a quantity, namely, a distance between the lines or surfaces containing it. This means that it would be a distance (or divergence) in one sense, although the angle is not on that account a straight line. For it is not everything extended in one sense that is a line. (Euclid, 1956, p. 177)

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[2]Substance, quantity, quality, relation, time, place, position, possession, passion, and action (Aristotle, 1980).

Carpus's definition certainly categorized an angle as a quantity, however the use of the word "distance" and "in one sense" was very puzzling for Proclus who was an early commentator of Euclid's *Elements*. At that time, lines had been considered "breadthless length," and therefore "magnitudes in *one dimension*" or "magnitudes extended *one way*," and when Carpus spoke of a "distance in *one sense*" between the two rays in an angle he was using words that until that time had been used only to describe the length of a line segment. He surely meant to indicate more of a rotational distance rather than a linear distance, but at that time, conceptions of magnitudes only allowed for measures in one, two, or three dimensions (i.e., length, area, and volume), and the vocabulary was clearly lacking to describe a rotational distance (Proclus, 1970).

This "distance" or space between the two rays can easily be misinterpreted because, if an angle's measure is held constant but you extend the rays indefinitely and travel out into the interior, away from the vertex, many measures are increasing. For example, if you progress away from the vertex and several lines are drawn across the angle connecting one ray to another, the lengths of these lines (one dimensional measure) will increase. As you travel away from the vertex, the rays are increasing in length as well. The area (two dimensional) is increasing too.

Past definitions that did not directly emphasize angles as quantities also posed problems in trying to identify the part that was to be measured. For example, Euclid's eighth definition in *The Elements* defined an *angle* as "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line"(p. 153). This definition categorized an angle as a relation because the emphasis is on the inclination of one ray to another. Proclus and others criticized this definition because if an angle is only one relationship between two rays, how can many angles exist for one inclination? For example, a 90° angle could be the same relation between two lines as a –450°, –270°, –90°, 270°, or a 630° angle. Therefore, this definition fails to distinguish these angles by their size created by the revolutions of the rays about the vertex. Rather, it suggests that if two angles are coterminal, then they have the same measure.

Mathematicians who posed definitions that emphasized the quality also had problems when it came to the angle's measure, because a quality was something like heat and coldness where you could have more or less of an angle…or one angle is more of an angle than another. Also a quality would not have the properties of equal or unequal or the potential to be divided into parts. (Proclus, 1970, pp. 98–99).

In the Classroom

Sixth-grade students' comments about angles also indicated confusion with angle size. Some of their initial definitions are very similar to the definitions from history. It was only after their ideas were shared and debated that other students or the teachers could point out the problems in interpreting the size of the angle that occurs when one aspect is emphasized more than another. These arguments are shared organized by the emphasis that seemed to cause the confusion.

Emphasis on quality. In one argument, some students were focusing on the quality aspect of the angle rather than the quantity when they made certain incorrect statements concerning a story that was read to the class. On the fourth day of the unit, one teacher read "The Greedy Triangle" (Burns, 1994). This story is about a personified triangle who gets greedy and wants to have more sides. A shape shifter comes along and turns the triangle into a square. The square then becomes restless and wants even more sides. So, as the story continues, the shape continues to change into a pentagon, a hexagon, heptagon, octagon, and so on. In the book, the picture of the first triangle was quite large and filled most of the page. So as the story progressed and the polygon gained more sides, the length of the sides of the *n*-gon were forced to decrease in length so that the shape could still fit on the pages of the book. However, as expected, the angles of each shape increased in size.

At one point in the story, the shape's "sides were so small it had trouble keeping its balance and it started to roll." The teacher interrupted here and asked, "Why did it start to roll?" Taylor responded, "Because its corners got smaller and smaller." The teacher inquired, "Its corners got smaller and smaller?" And together several members of the class chanted, "Angles!" At this point the teacher said, "Where's our original guy (the triangle)?" and led the class through the book looking only at the angle size. They soon were able to see that the interior angle was getting larger and larger each time the page was turned. [T2: Dec. 6]

This is a case where students were either confusing angle size with the length of the sides that, in this case, were getting smaller, or perhaps they believed that the length of the line segment connecting the outside endpoints of two adjacent sides was decreasing, which also was true in the majority of the book's drawings. However, a final conjecture could be, as one boy stated, that as the number of sides increased, "the corners of the shape were disappearing—they were less sharp, and therefore, less of an angle." If their confusion took on this last form, perhaps they were thinking of an angle more as a quality. Thus, as the story progressed the polygons' amount of sharpness was actually decreasing even though the interior angles were increasing.

Another confusion of this form occurred during the instruction when each student was given a set of regular polygons (referred to as a "shapes set"). Each of the regular polygons had sides of one inch. During one discussion Norm commented,

Norm: On the hexagon, the side, the angle comes to a sharper point than on the octagon.

T1: Okay. What can you say about the angle? A sharper angle, what do you mean by that?

Norm: It like sticks out further.
T1: Sticks out further. What do you think, is it bigger or smaller than the other one?
Norm: I think D (the angle in the hexagon) is sharper.
T1: Sharper. Does that mean larger or smaller?
Norm: Like it comes to…larger.

Norm's comment about the sharpness of the hexagon's 120° angle when compared to the octagon's 135° angle shows that he believes that a sharper angle is the larger angle. Norm's focus on the sharpness of the angle is reminiscent of the arguments mathematicians and philosophers of Aristotle's time who classified the angle as a quality. Those who thought it a quality argued that something could be more of an angle or less of an angle. If you consider the common everyday language that Norm is familiar with, such as "turning a sharp corner" in your car, or thinking of an "angular face" as one with a sharper or more pronounced bone structure, it makes sense that he would consider the sharper 120° angle more of an angle than the 135° angle. And perhaps, more of an angle, in his mind, is one that is larger.

Notice also, that in focusing on the sharpness of an angle, a student is really attending to the exterior angle (or possibly the reflex angle) rather than the interior. When you turn a sharp corner, your car has turned through the exterior supplementary angle even though the piece of land that is considered sharp forms the interior. This could also account for some of the confusion that the students were having in correctly interpreting the size of the angle.

Emphasis on quantity. Most of the students indicated that they thought of angles as quantities, inasmuch as angles could be measured in some way. However, some students differed in what portion of the angle was being measured. Some thought that the longer the rays, the greater the measure of the angle. For example, see Figure 1 for Brandi's entry in her journal. Notice that really all of the three angles she drew have approximately the same measure. Her understanding was that the quantity being measured was the length of the rays drawn.

Others thought that the more space between the rays, the larger the angle. This can be a very confusing concept however, as the next vignette highlights. One activity in the CMP unit focused on Amelia Earhart's last flight and how her navigator was off by about 7.5° from the correct route. A picture of a map was included with this small angle drawn on the page. The students measured the angle with their angle rulers, and then the teacher led them in the following discussion. (Asterisks separate different portions of the conversations.)

T1: About 7 ½°, okay. Was she off very much as far as the angle is concerned?

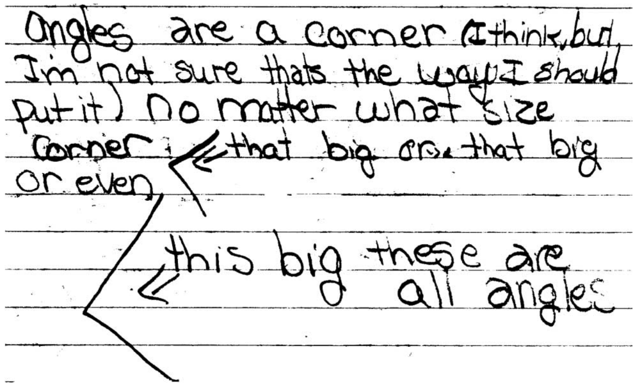

FIGURE 1 This journal entry indicates that the student was thinking that the length of the rays is the measure of an angle.

Class: NO.

Claire: But that's like on a small scale though. In real life it would be really big.

Wayne: It would be like 200°.

T1: What do you mean by that?

Claire: That's a small map. If you made the map bigger, if you blew it up bigger, it would be a different distance, because they wouldn't be so scrunched together. It would be farther apart. And if it was real life, it would be a really big angle. Like if you were actually measuring from a…

T1: You think so? Let me give you an example Claire. I made this triangle (pointing to a large equilateral triangle she had drawn on newsprint). I blew this triangle up. It's a lot bigger than the one on your paper. Do you still think these angles are 60° angles?

Claire: Well, the places would be farther apart because on that map, you have to scrunch things together a little more. You have to make them smaller.

T1: Lynn?

Lynn: [Lynn changes the conversation by talking about a 90° angle.] The farther you blow it up, the bigger your 90° angle would be.

T1: Okay…

Claire: The bigger what [would be]?

T1: Say that again?

Lynn: Like if you…Right now, it's a smaller 90° angle, then if you blew it up really big then the 90° angle gets bigger. So you have more space. So it would still be the same degrees, and that's your angle.

Claire: If it gets bigger, it wouldn't be 90° anymore.
T1: It wouldn't?
Claire: If it got bigger.

(Next day) The teacher is now emphasizing the word "turn."

T1: And again, when we measure angles, are we looking at the distance between the two rays? Or are we looking at the TURN?
Claire: Well if you weren't looking at the distance in the two rays, then well, there has to be, for a turn there has to be distance between them. So it wouldn't be an angle without the distance.
T1: But it wouldn't be an angle without the turn either. OK?
Claire: But the turn distance between there is, like, a lot bigger.
T1: A lot bigger in real life *(just repeating Claire's comment)*.
Claire: Yeah.
T1: Jack?
Jack: I've got a question for you (directing this question to Claire). In our shape set, you've got a triangle A and if you measure that and if you measure that (pointing to the large equilateral triangle hanging up in front of the class) how come they are the same angle? That one's bigger.
Claire: I don't know.
T1: Why don't we try that? Could you take triangle A out of your shape set. And can you take one of the angles and will it match up to the angle on my large one? We said that shape A was a 60, 60, 60. Alright, hold that there Linda (asking Linda to hold her small shape A up to the larger drawing hanging up in front of the class). Claire what do you think of that?
Claire: I don't know.
T1: So I can draw a 60°, 60°, 60° triangle very small with side lengths of probably 1 centimeter, 1 centimeter, 1 centimeter and I think that one is like 10 centimeters, 10, 10, but they each have 60° angles? Just something for you to think about.
Claire: But it just seems weird, like on a map if it's so big in real life and so small there, how the distance like in there can be the same.
T1: Jack?
Jack: It may be funny but it's true.

Claire really seemed to struggle with the idea that the measure of an angle would remain constant even though the length of the rays AND the space between the rays gets larger the further you travel out into the angle. Her struggles reiterate the problems already shared concerning the use of terms like "distance" when defining an angle.

This final vignette is an example of how some students emphasized the quantity aspect of the angle. One day in T2's class while the students were working on the assignment of finding 1/3 of a right angle turn, 2/3 of a 90° angle, ¼ of a right angle, and so on. June showed me a picture she had drawn and asked what she had done wrong. She had drawn the picture shown in Figure 2. June told me that she knew 90° was larger than 60° but in her picture, her 60° angle was larger. I asked her, "What is larger?" and she responded, "The angle is larger." I said, "Is it? Where is the angle? Which angle is opened apart farther?" She finally exclaimed, "Oh, I was looking at the arrows!"

Emphasis on relation. Not many of the students' comments fit into the category of "angle as relation," but the following conversation illustrates that at least one student objected to the use of angle being thought of only as the "width" or "distance" between two rays. His objections might suggest that he was considering more of the relationship between the two rays, and how they were "inclined" to each other.

[T2: Dec. 19]

T2: Okay, then what's the corner? Melissa, where is a corner? Is that where they meet together at that point? Can I measure a point?
Melissa: It's the width between the two lines.
T2: The width between the two lines. Tell me what you mean by that.
Melissa: They meet and like you can imagine the space in between them.
T2: You imagine this space in between. Do you agree?
Karl: No. Because, like, when you turn it all the way down to 270, you're not measuring the width between the two lines.
T2: You're not measuring this space in between?

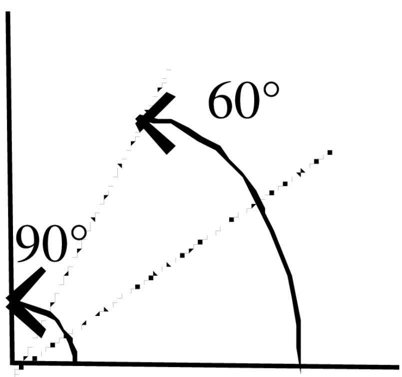

FIGURE 2 June's picture where she thought 60° was larger than 90°.

Karl: I'm not sure.

Melissa used the word "width" and the teacher later replaced her word with "space." Karl interpreted Melissa's comment as meaning a linear width or distance measured between two rays because he immediately questioned her definition in class. As long as Karl heard the word width, he objected to the idea, but when the teacher began to describe the angle as the space between two rays, he then became uncertain. Karl had a valid objection to using the word width because he believed it is impossible to measure a linear width between two rays if they are opened to a position larger than 180°. He used the example of a 270° angle that would look like Figure 3.

Karl's choice of words could possibly indicate that he viewed angles as relations. He said, "when you turn it all the way down to 270," which indicates that he was thinking more of how the two rays were related to each other. However, because he also was able to think of an angle as the quantity involved in a turning motion, he did not have the same confusion as introduced by Euclid's definition because, in his mind, he was rotating one ray away from the initial ray "all the way down to 270°."

All of these situations highlight how emphasizing certain features of an angle can cause confusion in interpreting the size of an angle. What is unclear when listening to students attempting to give a concept definition for angle by using a limited number of words is how their entire concept image is influencing their choice of words. Also, it is likely that the sixth-grade students in this study lacked adequate vocabulary to formulate clear descriptions for angle. Therefore, as the researcher, I attempted to make possible interpretations of what they might have been thinking, though these are just conjectures on my part. A future study should

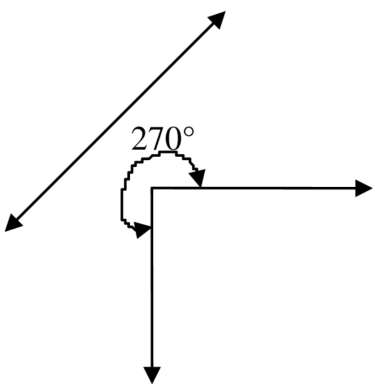

FIGURE 3 Example chosen by Karl to object to "width" definition of angle.

include interviews with students after classroom conversations to tease out their meanings more carefully.

CAN ANGLES CONTAIN CURVES?

The Historical Perspective

In the eighth definition of *The Elements*, Euclid defined an *angle* as "the inclination to one another of two lines in a plane which meet one another and do not lie in a straight line." The ninth definition continued, "*and when the lines containing the angle are straight* (italics added), the angle is called rectilineal" (p. 153). When Euclid specified that rectilinear angles contain straight lines, he was suggesting that other angles could be composed of lines that are not straight, or curved. More specifically, these curves were arcs from circles. So, for example, a horn-like angle was an angle that was formed by a line tangent to a circle and the circle itself (see Figure 4) or an angle could also be formed by two circles tangent to each other (see Figure 5; Proclus, 197, p. 102).

Proclus' (1970) commentary on Euclid's Elements provides a detailed classification of all kinds of different angles containing curves. His major objection to including curves in the angle definition was that if you propose that an angle was a quantity or a magnitude, and therefore something that can bear a ratio to another magnitude, then it is difficult to compare the magnitudes of angles containing curves (p. 98). So, for example, it is easy to compare the magnitude of a rectilineal 30° angle to a 60° angle by saying that one is half as large as the other. However, to compare the magnitudes of angles such as those in Figures 4 and 5 is a more difficult task. In other words, past objections to including curves were based on the difficulty in actually measuring such angles. For example, how would you go about measuring the angle formed by holding a taut string tangent to the earth's curved horizon?

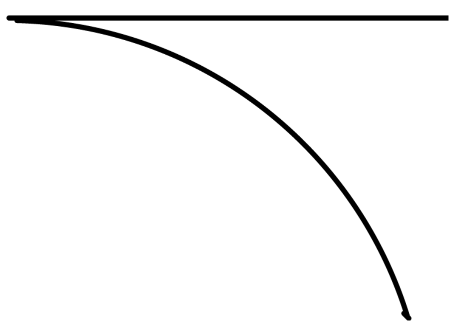

FIGURE 4 Horn-like angle.

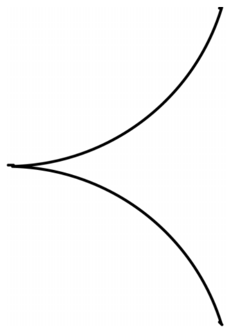

FIGURE 5 Angle formed by two tangent circles.

In the Classroom

Some students in the CMP classroom likewise considered whether curves could contain angles. In an early conversation when the students were discussing what kind of figures were polygons and that were nonpolygons, one student, Jenny, stated that semicircles contained angles.[3] It is evident from the following conversation that Jenny considered that a "strong bend" in a curve was also an angle.

(T1: Dec. 13)

Jenny: Doesn't cursive writing have angles? You know those cursive letters that have little wedges in them…Like the J. (She draws Figure 6 and points to two different locations on the letter.)
T1: I guess it depends on how you make the J. The capital J. I still think of that as being curved.
Jenny: There is also, (pause) … the little tail part.
T1: Right in there?
Someone: It all depends on how you make your J.

Cindi, a student in the other class, also brought up the question of whether angles were contained in certain letters of the alphabet. Her question sparked a lengthy discussion, part of which is given below, suggesting that many in the class were confused on this issue.

(T2: Dec. 18)

Cindi: Can you find angles in any letter of the alphabet? Except for "S" because it's kind of, well…maybe it does have angles! Because

[3]Of interest, the classification in the Proclus commentary included the angle created in a semicircle and classified it as a "line-convex," meaning that it was composed of the straight-lined portion of the semicircle and the curved portion was convex to the outside.

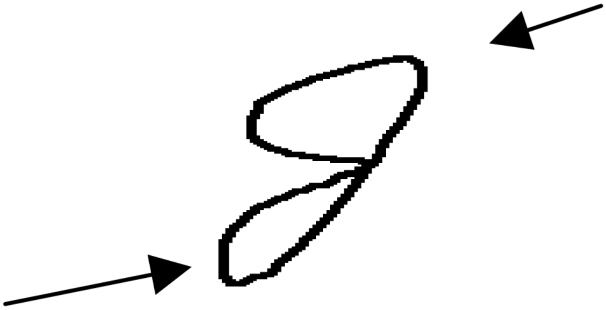

FIGURE 6 Jenny's drawing of the capital J.

you know how S's sort of have corners like here and here? (She draws an S up on the board and points to two places; Figure 7) So maybe it works for all letters in the alphabet?

Dan: Except O.

Cindi: Yeah, except O.

T2: Well, what do you think about these? (pointing to the letter S again)

Bruce: They are like corners, kind of.

Cindi: That's what I thought, but I don't know.

Karl: But according to, like, the little sheet that Mrs. Landis showed us, that's not an angle. (The little sheet was a handout that T2 had printed out concerning polygons, not angles.)

Cindi: I don't know.

June: If the S had part of the angle, then why wouldn't the O then, since it's round just like the S?

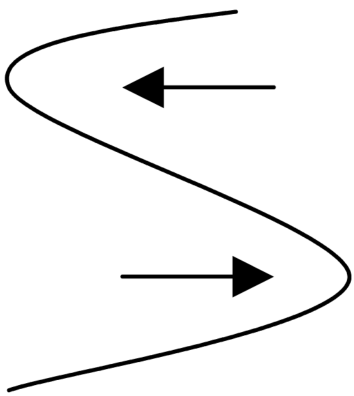

FIGURE 7 Cindi's drawing of the capital S.

Karl: Yeah, that's what I thought.

(More discussion concerning whether the S contains angles…)

Jamie: I don't think it does, because there is no place to meet together to make the corner of the angle. Because we said angles had to meet together with the other one… to make an angle.
Cindi: There is not, like, two lines in an S. There is only one line and it's just curved around. It's not like the R or P or something that two of them meet together.
T2: Okay, so what do you think?

(More discussion)

T2: Do we have one side meeting another side, forming a corner in the S?
Class: No.
T2: Are those angles in there?
Karl: No.

Notice that although the teacher eventually led the students around to her way of thinking that angles could not contain curves, her silence and small prompts to continue the students' participation drew out comments from several students who, like Cindi, were considering whether curves could form angles. Also, it is interesting that a few students believed that an R or P (written in uppercase) could have angles because there was an intersection of two curves or a line and a curve but that an O or an S could not because no intersections take place in those curves. This conjecture is similar to the angles in the classification contained in Proclus' commentary. These angles contained two parts such as line-line, line-circumference, or circumference-circumference. Thus, it is clear that the writer was describing the two parts that met at some intersection.

The previous comments indicate that it is not evident to all students that our current understanding of *angle* typically does not include curved lines. Discussions concerning the inclusion or exclusion of curved lines or surfaces may need to take place with students who are still forming their concept image for angle.

DIFFICULTIES WITH CONCEIVING OF 0°, 180° AND 360° ANGLES

As referred to earlier, thinking of angles as distances between two rays will exclude any angle greater than or equal to 180°. One student in this study really struggled to adapt her concept image for angle so that it could include specifically the 180° and 360° angles. This is not at all surprising given the number of definitions of angle from history that specifically exclude these angles that occur at the limits of our imagination.

The Historical Perspective

Consider the following definition: "the overlap (or intersection) of two half planes bounded by the arms" (Piaget footnoted this definition by Louis-Betrand (from F. Enriques, *Encycl. Math., III*, 1; Piaget, Inhelder, & Szeminska, 1960, p. 173). There would be no overlap or intersection of planes once the angle grew larger than 180°, so these larger angles do not exist given this definition.

Hilbert's *Foundations of Geometry*, first written in 1899 but revised several times until its tenth edition, defined angle in the following way:

> Let α be a plane and h, k any two distinct rays emanating from O in b and lying on distinct lines. The pair of rays h, k is called an angle and is denoted by $\angle(h, k)$ or by $\angle(k, h)$. (Hilbert, 1971, p. 11)

Notice that this definition excludes angles with measures of 0°, 60°, or 180° and as discussed earlier, given two rays with a common endpoint, it does not distinguish between an angle whose measure is x and an angle whose measure is $360°–x$.

Hilbert's *Foundations* have strongly influenced geometry texts in the United States. For example, in the preface of *Euclidean and Non-Euclidean Geometries,* a popular college level text still in use today, Greenberg (1993) stated that he used modified versions of Hilbert's axioms. His definition for angle is also very similar to Hilbert's:

> An "angle with vertex A" is a point A together with two distinct non-opposite rays $\overrightarrow{AB}$ and $\overrightarrow{AC}$ (called the sides of the angle) emanating from A. (p. 17)

Greenberg explained in a footnote that he eliminated the possibility of a "straight angle" because "most of the assertions we will make about angles do not apply to "straight angles (p. 17)." Other college texts (Coxeter, 1961; Fishback, 1962) have used Hilbert's definition almost verbatim.

In Allendoerfer's article concerning a proper definition for an angle (1965), he wrote,

> A simple angle is the union of two noncollinear rays having the same origin. I have excluded collinear rays because there is trouble in the definition of the interior of an angle whose rays are collinear. If the rays are not only collinear, but identical, the interior can be taken as the empty set and the trouble is resolved. If we wish, we can call such a configuration the "zero angle" and include it among the simple angles. But if

> the directions of the two collinear rays are opposite, we have no good way of deciding which half-plane to choose as the interior. Nevertheless, we call such a figure a "straight angle" and consider it to be somewhat distinct in nature from simple angles. (p. 83)

All of these writers have clarified many of the problems that can occur when trying to include angles with measures of 0°, 180°, 360°, and so on.

In the Classroom

Claire also excluded a "straight angle," and she had many reasons to support this exclusion. Likewise, the 360° angle did not fit into her concept image of angle. Claire repeatedly raised questions about these special angles over the span of several days. She raised so many questions in class that I decided to interview her specifically to learn more about her conceptions. Her comments show how she relies heavily on her everyday experience with the physical world as she attempts to make sense of mathematical concepts. When she could not find a practical example or application that made sense to her, she had considerable difficulty understanding the concept. Her first question in class was as follows:

> I have a question about a 360° angle. Isn't an angle like a point on something? I'm kind of curious about 180° too. But 360°, all it is, is a circle. So I don't really think…so I am kind of wondering why 360° and 180° are considered angles?

In the following paragraphs, I have listed several of Claire's comments that she made in class and in the interview concerning her image of angle. They are categorized by comments made specifically about 180° angles and 360° angles, and then angles in general.

180° comments:

"If that's an angle, then it needs to have two sides."

"The same with 180°. It's like … I don't see where the two lines connect."

"Because I saw an angle as having a vertex point and then two lines going different ways. And I know they are going different ways [in a 180° angle], but you can't tell where the vertex is and how they are going different ways or if they're sort of doing the same thing."

360° comments:

"Isn't an angle like a point on something?"

"Well, if you draw an angle,… when I see angles, like around the house or around the city or something, they don't have the other lines in it, on the beginning part." (I believe she meant that the other ray is not on top of the first ray.)

"…a circle doesn't have any angles. So I don't understand that."

"But if you see a circle somewhere, it might not have something inside, and I don't see how that can be angle, because an angle has to have …like an angle can be like… an angle is just a point, and there is no point in a circle."

"Angles are lines. I don't think of them as a motion, I think of them as lines."

"And I don't understand how that can be the corner because there is nothing touching it. Unless this is solid, well, the inside of an angle I don't think is really solid…it's just the lines itself. But on this one, I don't understand it that way, how it could be an angle."

"And all I think what the angle really is, is two lines put together on their corner, propped up on their corner, but this has no corner to be on. I don't understand how an angle can just be a turning motion and not two lines."

Angles in general:

"It's just the sides, how they're put together on the corner."

Claire's comments indicate that she really struggled to take a physical situation and abstract from it a 180° or 360° angle. Her practical everyday experience taught her that angles are useful in shapes, but she really struggled with the turning aspect of the angle and therefore found it difficult to conceptualize these angles. Perhaps if she had had more opportunities to explore the rotational idea of an angle she would have been able to add these angles to her concept image.

Having looked for several definitions for *angle* from history, I found very few that focused primarily on this rotational aspect. One definition that did is as follows:

> For every [point] $O \in \pi$ [plane], a rotation about O is called an angle with vertex at O. If (A_1, A_2) is a pair of half-lines whose origin is O, the rotation about O taking A_1 to A_2 is called the angle formed by the pair; it is written A_1A_2. (Choquet, 1969, p. 79)

Choquet's definition included zero, straight, and angles larger than 180°, but angles were only distinguishable mod 360°. It seems that this particular definition could be beneficial to students in that it helps to explain 0°, 180°, and 360° angles.

CONCLUSION

Perhaps if more textbooks, like CMP, included multiple representations for *angle* rather than presenting one limited definition, students would have more success in understanding its multifaceted nature. U. S. textbooks from the past 20 years most typically provide a static definition for angle. This approach allows for the concept to be introduced more quickly but it robs students of opportunities to experience angles as rotations that might affect their experiences in trigonometry and other advanced mathematics courses that require some flexibility in this area. The Dutch mathematician and educator, Freudenthal (1973), expressed his concern about using only one definition for angle in his book *Mathematics as an Educational Task*. He stated,

> As has been stressed several times, there is more than one angle concept. Some didacticians claim that there is only one that is correct. Love of order is fine unless it goes as far as to forbid important concepts because they do not fit into the system. Properly said such would be a bad mathematical attitude. It has cost a great deal of trouble to get mathematicians used to the fact that there are various number concepts, which are now carefully distinguished from each other. If rather than being distinguished all angle concepts but one are forbidden, pupils will never learn to distinguish them—forbidding rules never work. (p. 476)

It should be evident that throughout history the geometric term "angle" has been formally defined in many ways. Some definitions refer more to its rotation, its interior, or some more to its composition of rays and a vertex. Some definitions exclude angles with measures of 0°, 180°, or 360°, or all angles larger than 180° or larger than 360°. Many of the past definitions allowed for curved portions that intersected in some way. All of the definitions could be classified as either a relation, a quantity, or a quality. The many definitions stem from the fact that the concept of angle is highly complex and can be approached from a variety of vantage points. If history is inundated with examples of the complexity of the angle concept, then there may be much that students can learn from the struggles that others had in trying to capture its meaning.

Another very important feature that allowed the students of this study to broaden their conceptions of angle was that their teachers provided many opportunities for discussions, arguments, questions, and concerns to be raised. The teachers' patience in giving the students over 5 weeks to ponder the question, "What is an angle?" allowed the students to interact and challenge each others' thoughts. At the same time, the teachers did have some influence over the course of these discussions and could think of ways to challenge the students when their thinking had not been well-informed. If a student emphasized one feature of an angle so that it clouded his or her understanding of a different entity, the teachers often provided counterexamples that would challenge the student's conception. All of these interactions contributed to the students' learning of the concept.

Most mathematicians today could probably provide a sound mathematical definition for angle. However, depending on their specific mathematical specialty, the meaning may take on a very different emphasis as it is applied to that field. That is, a geometer may define "angle" in a much different way than would a topologist. Therefore, to help middle-school-aged children understand the concept of angle, this study suggests that presenting angle using multiple representations would seem to be more comprehensive and a better preparation for higher level mathematics. It also suggests that when a concept is as complex as *angle*, permitting students to share and challenge each other's ideas helps students develop a more complete concept image—one that can be applied to many different situations.

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