How do we Provide Tasks for Children to Explore the Definitions of Quadrilaterals?

Alice Hansen
St. Martin's College
<a.hansen@ucsm.ac.uk>

Dave Pratt
Warwick University
<dave.pratt@warwick.ac.uk>

This paper considers task design related to the geometrical understanding of a class of nine- and ten- year old children. The research was interested in identifying the principles underpinning these tasks. A first iteration involved a nationally-recognised sorting task (National Numeracy Strategy, DfEE, 2000) and a second principled task was designed to delve further into the children's understanding of the inclusive nature of the definitions of quadrilaterals. We consider to what extent van Hiele's levels of geometrical understanding can be used at the classroom level and raise the issue of appropriate tasks for children to engage with in order to challenge and stimulate their understanding of geometric definitions.

Understanding of geometric definitions is a complex area to study. Let us offer three putative reasons why this may be the case. Our environment has a great impact on the way children perceive shapes and it is in this environment that children are continually exposed to shapes that have a horizontal base. An alternative argument could be the traditional use of paper-based technology. Both create prototypical shapes that lead to visual cues being predominantly used by children. Even when we remove the horizontal base and rotate a shape, to say a kite flying, we still view a prototypical shape (Hershkowitz, 1990).

Another aspect of geometric definitions that compounds the complex nature of the topic is that of the inclusivity and exclusivity of definitions. Children find it difficult to consider that a square is a rectangle (demonstrating the inclusive nature of the definition of rectangles) (Jones, 2000), particularly when they are faced with images in and out of school where what they call a rectangle is, again, a prototypical oblong (illustrating the exclusive nature of the definition of oblong: a rectangle that is not a square).

Finally, the large number of attributes that can be used to define shapes adds to the confusion. For example, we can consider the length of the individual, parallel or adjacent sides, the size of the interior angles, the order of rotation and the lines of symmetry.

This paper considers how task design can begin to meet the needs of children who are struggling with these concepts.

Theoretical Framework

van Hiele's (1986) levels of geometrical understanding provide an overview of children's geometrical understanding, which is often used in studies in this domain. For example, Sutherland, Godwin, Olivero and Peel (2002) are using Dynamic Geometry software (Cabri) in their study concerned with the process of developing a design initiative for primary children to learn about the properties of polygons, drawing explicitly on the role of the teacher exploiting the dynamic nature of Cabri. They are particularly interested in the children becoming aware of the invariant properties of particular quadrilaterals. Sutherland et al. found that the majority of children in their studies recognised shapes by drawing on the strong visual clues as described in van Hiele's level 1. They wish to develop the children's mathematical understanding across van Hiele's levels 1 and 2.

However, we think of van Hiele's levels as belonging to a family of macrolevel theories, which describe the nature of mankind's knowledge as it evolves over long time scales, a trait arguably begun by Piaget in his work on genetic epistemology. Our focus is however much more humble. We offer the reader a metaphorical microscope with which to zoom in and in until we see, not the knowledge of mankind evolving over a life span or beyond, but the meanings that an individual child is struggling to construct at a moment in time and how those meanings change over hours or even minutes. What you might see through this microscope is not beautifully organised and smooth levels of achievement but instead noise and inconsistency. We see a child whose knowledge is in a state of flux and under constant pressure from outside influences. Of those many structuring resources in the classroom setting on which we are now focussed, we ask what the contribution of the task is to this complex and excitingly unsmooth dynamic.

van Hiele gives a relatively minor role to *intuition* in his levels (Fujita & Jones, 2002), and yet we see intuitions as particularly relevant to the immediate meaning-making that takes place at the micro-level. Fischbein (1994) makes explicit the place of intuition in geometry. He explains that "the interactions and conflicts between the formal, the algorithmic and the intuitive components of a mathematical activity are very complex and usually not easily identified or understood."

Fischbein (1993) blurs the edges of van Hiele's levels and reiterates the complex nature of geometry by observing that a geometrical figure "possesses a property which usual concepts do not possess, namely it includes the mental representation of space property". Fischbein argues that all geometrical figures are characterised by the interaction between their figural and conceptual aspects, leading to the notion of 'figural concepts'. He explains that with "age and the effect of instruction… the fusion between the figural and the conceptual facets improve".

There are few guidelines as to what this instruction could be and more specifically, how geometrical tasks could aid children in developing their intuition. Freudenthal (1981) offers a suggestion for how a familiar environment may lead to children's understanding of geometry. From a very young age, before a child is able to articulate their thinking, they are able to grasp space and relations in space by 'seeing, listening and moving in space'. The child undertakes the process of becoming conscious about their intuitive grasp of space and during this time verbalisation also occurs, leading to definitions, theorems and proofs. What Freudenthal is unspecific about, however, is *how* a teacher can encourage a child to develop definitions of shapes in this process. Indeed, although geometry is now officially included in the Dutch mathematics curriculum, it is not an area that has been completely implemented into present classroom practice (van den Heuvel-Panhuizen, 1998).

Methodology

This paper presents part of a wider research project, which sits within the design experiment paradigm (Cobb et al., 2003). Through following a series of features inherent in all design research, Cobb et al. (2003) propose that design experiments develop "relatively humble" theories that target domain-specific learning processes about "both the process of learning and the means that are designed to support that learning". These theories are built in a rather pragmatic way, taking an existing theoretical framework and, through a highly interventionist experimentation period involving iterative design, new theories are developed through the design and redesign of a conjecture, a product. It was through the classroom-based use of a product (which in this case was a design for a task) within the iterative process that children's definition of quadrilaterals was explored with the aim that we would first be able to abstract principles related to the design of a task about geometric definitions and subsequently propose more generic principles for task design.

Within this paper one iteration is discussed and principles for another iteration are considered. Within each iteration there were five sections:

(i) A design based on the conclusions from the previous iteration analysis;
(ii) A specific method;
(iii) The data collection process;
(iv) Analysis of the data and discussion; and
(v) A conclusion drawing out recommendations to be fed into the next design.

Iteration 0 was different only in so far as step (i) was a bootstrapping analysis based on an already existing nationally-recognised sorting task from the National Numeracy Strategy (DfEE, 2000a) in order to identify the principles which would underpin Iteration 1. Iteration 1 involved a second task that was designed to provide some insight into the children's understanding of the inclusive nature of the definitions of quadrilaterals, thus further developing the principles.

In preparation for a design experiment, many assumptions must be made about the starting points, elements of trajectory and prospective endpoints of the learning (Cobb et al., 2003). To avoid solely playing a 'mind-game', we used a software task [Carroll Diagram, DfEE, 2000b] recommended by the National Numeracy Strategy, which in England sets out the teaching framework from age 4 to 14; nearly all teachers in the state system follow this framework. In light of the activity on this task, we identified the need to provide a purposeful task that offered the opportunity for children to access a wide range of examples that could be manipulated, and which allowed for exploration of the properties of quadrilaterals and the relationships between them.

We built these ideas in to the Iteration 1 task (Figure 1), a hands-on classification task encouraging children to identify the attribute of the sets in an unlabelled Venn Diagram. The children had access to strips of cardboard, scissors, Blutak, a protractor and a ruler. They were asked by their teacher to create quadrilaterals using any of the resources and to place them in the blank Venn Diagram. Once placed, the teacher revealed to the children whether a quadrilateral had been correctly placed. If not, the children chose whether to adapt the shape or move it to a different set, and again received feedback from the teacher on the accuracy of placement. The children were encouraged to create as many quadrilaterals as they felt were necessary in order to classify each set. The game concluded when the children had labelled all sets. Four pairs of children were selected from the previous iteration to represent a cross-section of attainment. Each pair was video-taped and analysis was undertaken according to how the children constructed the quadrilaterals, how they were orientated and how the children perceived classification.

Construction was expected to be undertaken using any of the following methods:

- *Arbitrary* cutting of a strip/strips to make sides;
- Cutting a strip using some *visual* clue (no physical comparison with another side);
- *Comparison* made, either cutting more than one piece alongside another; folding a strip into half or quarters; holding an existing side against a new strip and cutting the new strip;
- Using a *measurement* tool (ruler or protractor) to assess the size of side or angle needed.

The methods of construction were used as a means of focussing progressively (Robson, 1993) on the children's understanding, thus informing our stance towards the design of the next iteration task. In this paper, we focus on further insights gained from Iteration 1, which we are embedding into the task for Iteration 2.

Figure 1: The iteration 1 task, Get Sorted.

Findings

It became clear from analysis of the data in Iteration 1 that there were several sources of confusion about the nature of geometric definitions. We set these out below.

Instance vs. Class

Several of the children did not identify how different instances of the same class could occur. Here, Katie and Callum discuss how two oblongs could sit within one set:

Katie: But then it will be like that one [motions to existing rectangle] though, but smaller. That would be good. What would you call it though? Small rectangle. *Continues to make 'small rectangle' by comparing and cutting strips.*

Callum: Yeah, but we could have it standing up.

Katie: Yeah, that way or that way?

Callum: No, keep that like that.

The need to label each instance, rather than the class to which several instances belong, appeared to be part of a more wide-ranging difficulty for the children in distinguishing between instance and class.

Attributes

The children were concerned with only the basic attributes of the shapes of right angles and the length of sides. For example, Luke and Tom used their knowledge of fractions to create the sides of a square:

L folds one strip in half.

Luke: We need one more [strip].

T folds one strip in half.

Tom: So we get a square?

Luke: A square … and we can do them into quarters.

L and T create a square together.

Luke: Get the quarters and put them all together.

The children used only *comparison* or *visual* construction techniques when considering attributes. None used the measuring resources (ruler or protractor) provided. Whilst the length of sides and right angles are important attributes, we believe that other attributes such as the size of the interior angles (other than 90^{o}), the order of rotation and the lines of symmetry are also necessary aspects of geometrical understanding. More generally, we saw much evidence to confirm Fischbein's view that there is a tendency for children to focus on perceptual rather than figural aspects of geometric shapes. These findings endorse Sutherland et al's (2002) research.

Inclusivity

None of the children appreciated the inclusive nature of definitions; they perceived the sets of the Venn Diagram as discrete and incorrectly labelled the sets similarly:

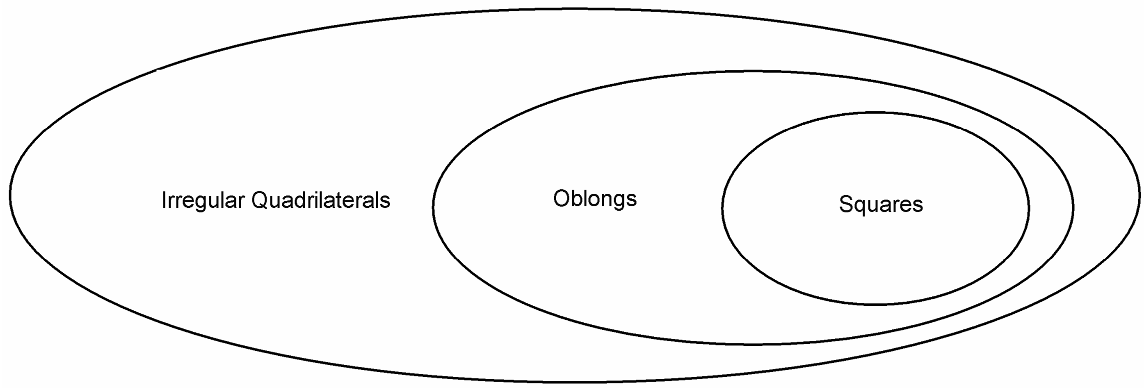

Figure 2: Typical incorrect labelling of Venn Diagram, Iteration 1.

James and Vinny's discussion is typical:

J takes label and writes OBLONGS, places it in rectangles.

Vinny: Squares in there. *J takes label and gives pen and label to V who writes SQUARES*

James: That's quadrilaterals with no right angles.

Vinny: I'm not sure. *Both laugh.* Don't do it as big.

James: What could they be? [pause] What would you call them shapes?

Vinny: No right angles.

James: They're all quadrilaterals.

Vinny: Yeah.

James: Quadrilaterals that have no right angles. *J writes it on a label*.

In general, we saw considerable evidence in support of the literature that has shown the difficulty that children have with the nature of inclusive definitions (Jones, 2000).

Defining

The task did not stimulate the children to define shapes. Rather than making use of the scope for manipulation built in to the design of the task, the children created predominantly prototypical instances, moving them between sets to gain a correct placement rather than persevering with the class in each set. This resulted in Luke and Tom incorrectly completing the task very quickly, placing only one shape in each set before naming the sets and completing the task. Harry and Rebecca also placed one shape in each set before tentatively naming them, but decided to confirm their (also incorrect) thinking by producing more prototypical shapes:

Harry: *Pointing to each subset.* They're all irregular shapes, they're all rectangles and they're all squares.

Rebecca: We should put some more in, just to check.

We were aware that the task for Iteration 1 failed to put children in the position of being definers as exhorted by de Villiers (1998).

Conclusion

The *modus operandi* of design research dictates that insights gained from the previous iteration are transformed into design conjectures, embodied in the design of the next iteration. We have not yet reached the full embodiment of the task for Iteration 2. However, we are able to indicate how the above findings are being operationalised:

- In response to the difficulty children encountered in distinguishing between class and instance, we intend to offer access to a range of instances of a definition that cover the whole scope of the definition. We considered the approach used in Dynamic Geometry where the child is able to drag a figure through many instances with the potential that the child may abstract the invariant aspects of the construction. However, we feared that, when not in dragging mode, the child would once more be drawn by the specific instance in front of them. We therefore propose to adopt the use of animations that loop continuously through the whole panoply of relevant instances.
- Recognising that children usually fail to attend to the attributes of a shape, we propose to draw on the *Constructionist* (Harel & Papert, 1991) tenet that technology facilitates the construction of knowledge through use of that knowledge (see the *Power Principle* in Papert, 1996). Hence, we aim to present the essential attributes as tools with which the children can build definitions.
- The inclusive nature of some mathematical definitions also created problems for the children. One problem is that definitions are to some extent arbitrary. Not all mathematical definitions are inclusive and it is very easy to generate arguments even amongst experts (perhaps especially amongst experts!) about whether certain shapes are special cases of others or not. We see, however, a helpful affordance of microworlds with respect to this issue. We believe that, by building the normalised

definitions into the model underpinning the microworld, the task can be so designed as to involve the children in discovering what the computer knows.

- We are searching for a task design that generates purpose leading towards the construction of utility for defining (see Ainley, Pratt and Hansen, in press, for a full discussion of the two constructs, *purpose* and *utility*). We believe that by offering the attributes as tools for building, we have a foundation for designing a purposeful task, but at the same time recognise the non-triviality of such an aim.

We have taken one further step in the process of operationalising the conjectures emerging from Iteration 1. We are referring to the animation of instances to scope the definition of a class as *might-bes*, in the sense that an instance of the definition might be represented by any of the frames presented. *Might-bes* are a series of morphing instances that cover the whole scope of a definition. In Figure 3, we give an example of a series of might-bes for parallelogram:

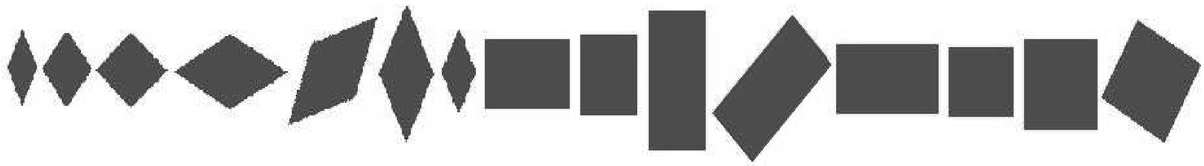

Figure 3: Example of a series of *might-bes* for parallelogram. Each element of this figure represents one frame in an animation that proceeds in time from left to right.

We intend that *might-bes* should be built from attribute-based tools, which we have labelled *must-haves*, in the sense they are attributes that are requirements within the definition. For example, one such building block would be “must have at least one right angle”. The phrasing of the *must-haves* intentionally foregrounds the inclusive nature of the definitions. By offering these attributes as building blocks, we expect the children to engage playfully with them, and construct new understandings through their use (see Figure 4).

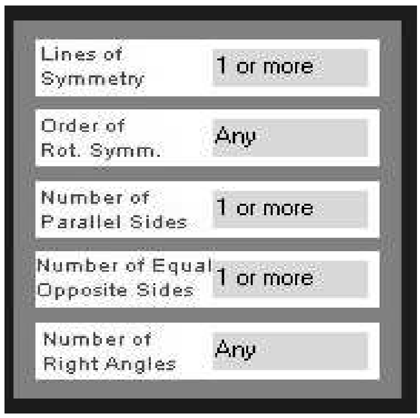

Figure 4: Example of *must-haves*. The student is able to change the minimum setting for the corresponding attribute before using these settings to create the might-bes animation.

With these specific embodiments now in place, we are close to finalising the task for iteration 2. We will be collecting data for iteration 2 during June 2005. As a result, we aim to offer specific guidelines about the principles for task design with respect to geometric definitions as well as at a generic level.

Acknowledgement

We would like to thank Anchorsholme Primary School, Blackpool, UK for their support in undertaking this research.

References

Ainley, J., Pratt, D., & Hansen, A. (in press). Connecting engagement and focus in pedagogic task design. *British Educational Research Journal*.

Cobb, P., Confrey, I., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher, 32*(4): 9-13.

Department for Education and Employment. (2000a). *Using ICT to support mathematics in primary schools.* London: Qualifications and Curriculum Authority.

Department for Education and Employment. (2000b). *Carroll Diagram*. Accessed 20 March, 2005. http://www.standards.dfes.gov.uk/primary/publications/mathematics/12874/nns_useict026000carroll.swf

De Villiers, M. (1998). The evolution of pupils' ideas of construction and proof using hand-held geometry technology. In A. Olivier & K. Newstead (Eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology Mathematics Education* (Vol. 2, pp. 337-344). Stellenbosch, Sth. Africa: PME.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics*, *24*, 139-162.

Fischbein, E. (1994). The interaction between the formal, the algorithmic and the intuitive components in a mathematical activity. In R. Biehler, R. Scholz, R. Stässer & B. Winkelmann (Eds.), *Didactics of mathematics as a scientific discipline* (pp. 231-245). Dordrecht: Reidel.

Freudenthal, H. (1981). Major problems of mathematical education. Educational Studies in Mathematics, *12*, 133 – 150.

Fujita, T., & Jones, K. (2002). The design of geometry teaching: Learning from the geometry textbooks of Godfrey and Siddons. In O. McNamara (Ed.), *Proceedings of the Twenty-third Day Conference of the British Society for Research into Learning Mathematics* (pp. 13-18). London, UK: British Society of Research into Learning Mathematics.

Harel, I., & Papert, S. (1991). *Constructionism*. Norwood, NJ: Ablex Publishing Corporation.

Hershkowitz, R. (1990). Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and Cognition* (pp. 70-95). Cambridge: Cambridge University Press.

Jones, K. (2000). Providing a foundation for deductive reasoning: Students' interpretations when using dynamic geometry software and their evolving mathematical explanations. *Educational Studies in Mathematics*, *44*, 55-85.

Papert, S. (1996). An exploration in the space of mathematics educations. *International Journal of Computers for Mathematical Learning, 1*(1), 95-123.

Robson, C. (1993). *Real world research*. Oxford. Blackwell.

Sutherland, R. Godwin, S., Olivero, F., & Peel, P. (2002). Design initiatives for learning: ICT and geometry in the primary school. In O. McNamara (Ed.), *Proceedings of the Twenty-second Day Conference of the British Society for Research into Learning Mathematics* (pp. 73-78). London, UK: BSRLM.

van Hiele, P. M. (1986). Structure and insight: A theory of mathematics education. London, UK: Academic Press Inc.

van den Heuvel-Panhuizen, M. (1998). Realistic mathematics education, Work in progress. In T. Breiteig & G. Brekke (Eds.), *Theory into practice in mathematics education.* Kristiansand, Norway: Faculty of Mathematics and Sciences, Hogskolen I Agder.