THE USE OF EXAMPLES TO PROVIDE REPRESENTATIONS IN PROVING

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To contribute to debate about the need and nature of example generation in the proving process, we analysed videos of students working on different proving problems. Our analysis was based on integrating three frameworks: achieving conceptual insight and a technical handle while proving; manipulating, getting-a-sense, and articulating in mathematical work; and the use of syntactic and semantic modes of proving. A key aspect of successful use of examples in proving is to alight on a representation, or a coordination of representations, that provides both conceptual insight and a technical handle. We illustrate this finding in one problem.

INTRODUCTION

It is our personal experience that the exploratory creation of examples can help towards constructing proofs. This practice is supported by Pólya (1962) and many others. Nevertheless the notion that students could be usefully advised to do this has recently been challenged by Ianonne, Inglis, Mejia-Ramos, Simpson, and Weber (2011, p.13) whose experiments concluded that there was not enough evidence for example generation to be 'a viable pedagogic recommendation'. This apparent contradiction led us to focus not on example generation as an imposed strategy, but on how students who had experienced the use of examples used it in their proofs.

In our previous work, we analysed video data from 27 university students working in groups on different proving problems, aiming to understand the contribution made by example generation when used naturally (i.e., as a tool that is available to the prover) in the proving process. In Sandefur, Mason, Stylianides and Watson (forthcoming) we identified and illustrated the following four aspects of situations in which example generation has a positive role to play in proving. These aspects conjoin qualities of students and of problems:

(1) *Experience of utility of examples in proving*. Students have experience of constructing examples and are disposed to do so (e.g., they know how examples can expose structural relationships).

(2) *Problem formulation*. The problem does not point directly to a productive direction for its solution (e.g. it might be phrased 'prove or disprove' or might require reformulation).

(3) *Personal example spaces*. Students' personal example spaces include appropriate familiar objects and methods, which can display underlying relationships.

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(4) *Relational necessity*. The combination of the problem and the students' resources are such that it is necessary to attend to underlying relations, and cannot be completed by only manipulating symbols.

Our analysis also suggested that a key feature of successful use of examples in proving is the selection of representation, which we discuss in this paper.

EXAMPLE CONSTRUCTION AND USE

There has been significant research into how students and mathematicians generate examples. Dahlberg and Housman (1997) noticed that students who exemplified widely and visually did gain a better understanding of a new concept than those whose actions were more limited. Iannone *et al.* (2011) extended this work with a large number of undergraduates from different universities. In one study they compared the success rates for proving statements of students who had been prepared in two different ways: one group through generating many examples of the central concept, the other by reading worked examples. Neither group knew in advance what would follow these activities. The authors found no significant differences between the two groups, neither of which seemed to benefit more than the other from the specific preparation in using examples. However, in these studies exemplification was imposed by the researchers. Methods of example generation and use may be different if exemplification is a tool available for use in order to achieve a mathematical purpose (Watson & Chick, 2011). Proving behaviour has been described as either syntactic or semantic. 'Syntactic' means the manipulation of symbols within the given representation system; 'semantic' is indicated by the introduction and use of other representations, which would include exemplification. Alcock and Weber (2010) suggested that semantic use of examples has four conditions for success: the prover can exemplify; examples are 'correct'; examples relate to formal definitions; and examples suggest inferences. The first three of these are subsumed in our third aspect above; the last we see in our fourth aspect, that is in interaction between the problem and the solver. The representation system in which the problem has been presented is also likely to have an influence on the approach to proof.

To summarise, there are few studies of how people spontaneously incorporate example use into purposeful mathematical work and those that exist suggest that it is not only the problem, and not only the disposition of the prover, that influence whether examples might be used or not.

METHOD

We used a body of videos of students working in groups of two or three to produce proofs which arise in an 'Introduction to Proof' course. The second author has worked with others to create an online video-library for use as case studies, to be discussed in courses on proof (Birky, Campbell, Raman, Sandefur, & Somers 2009) (NSF grant #1020161). The students were either taking a basic 'Introduction to Proof' course or were advanced students who had taken this and some higher-level mathematics courses. They have all been introduced to a variety of proof techniques, including deduction and proofs by contraposition/contradiction/mathematical induction. They have seen and experienced the need to engage in concepts rather than merely manipulate symbols in proving. The proofs produced in all the videos we analysed were correctly reasoned, but some were incomplete.

ANALYTIC APPROACH

We selected and analysed 11 videos of students working on three problems with sufficiently different approaches to warrant closer attention. In this paper we use extracts relating to one problem to illustrate our analytic approach, but our final conclusion is based on the full set of videos. We viewed the videos several times separately and together, having extensive discussions about the ways in which students used examples. We compared their actions and words to three established frameworks using a cyclic process of analysis, refinement and re-analysis which tested the frameworks and data mutually against each other. The frameworks were:

MGA (Manipulating; Getting-a-sense-of; Articulating): MGA integrates ideas of Bruner (1996) into a spiral of activity during mathematical thinking (Mason, Burton & Stacey, 1982). The manipulation of mathematical objects includes manipulation and inspection of examples to 'get a sense of' underlying structure and relationships. As that structure gradually becomes more coherent it can be articulated.

S/S (Semantic/Syntactic): We intended to distinguish between working within the symbolic system in which the proof statement is made (syntactic), and stepping outside the symbolic system (semantic) such as considering examples (e.g. Alcock & Weber, 2010). From our experience we knew this distinction can be problematic. It is possible to work formally and correctly with symbols and not consider underlying concepts, or for the symbolic form to be used and understood as a conceptual embodiment. It is also possible to use the same representation procedurally or meaningfully at different stages of proof.

CI & TH (Conceptual Insight & Technical Handle): Birky *et al.* (2009) (based on ideas of Raman (2003)) suggested that an important component in proving is recognition of the key idea in a problem. They observed that sometimes students gain CIs into the key idea but do not have access to THs with which to reason, and sometimes students have access to THs but have no CIs to direct their use. We extended this to fit our early observations of prover behaviour and identified a need for the prover to: (1) gain CI that indicates why the statement is likely to be true, and (2) find TH to convert CIs into acceptable proofs.

PROBLEM: FUNCTION COMPOSITION

We now illustrate how we used these frameworks to describe the work of an advanced pair of students (referred to as students 'Pip' and 'Sam ') on one problem.

> Given that g and h map A to B and that $fog = foh$, prove or disprove the following: (1) If f is onto from B to C, then $g = h$. (2) If f is one-to-one from B to C, then $g = h$.

Part 1 is false and requires a counterexample while part 2 is true and requires a proof. A useful conceptual insight would be for the prover to understand the distinction between'one-to-one' and 'onto'. Plausible technical handles would depend on what students had previously found useful in manipulating function composition.

Students' Work on Part 1

Pip and Sam were first given part 1. They wrote the problem down. Pip said: "My first instinct is to construct an example.... Maybe we should write down what onto means." They then drew Figure 1 and stated that this cannot happen if *f* is 1-to-1.

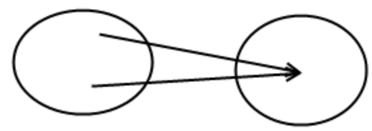

Figure 1: Understanding definition of one-to-one.

They wrote down the definition of 'onto' but did not give a specific example, unless we regard the diagram as an example of a way this could be represented. Then they wrote $f(x)=x^2$ mapping R to R. They also wrote that *g* and *h* are from R to R.

Sam : We have to show they are the same if their composition is the same.

Pip: Our example is too complicated. …Is your instinct that it works?

Sam : Yeah.

Pip: Yeah, my instinct too, I don't know if it's true, but here is what I'm thinking, let's say that we have A (draws circle as in Figure 2), we've got B (draws and labels circle B) and we've got C (draws and labels C), okay, now, I'm thinking, (unintelligible) for a second, … all we know is, we know, we know, *f* from B to C, (pauses with pen ready to draw from B to C) I think it's false (pulls pen back), let me show you why (moves pen back toward figure) if, let's say this is *f* (writes *f* between B and C) and it takes this point to C (plots point in B and draws line to C) then all that it's saying, as long as *g*...

He then drew *g* going from x_1 and *h* going from x_2 in A to point F in B, as in Figure 2. After a little mumbling by both, Pip said "but these are different *x*'s."

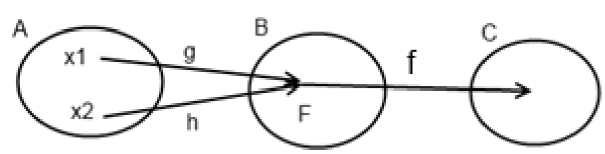

Figure 2: Initial attempt at understanding composition.

Sam then drew the arrow from x_2 to a different point in B and from that to the same point in C. After a little more discussion, Pip said: "What you drew shows that it's not true. We don't know anything about *g* and *h*, so imagine, we can say that *g* and *h* aren't 1-1, something like that, right, so imagine whatever crazy we can come up with, let's say that *g* and *h*, they have to both act on the same *x* and let's say *g* maps it to one point and *h* maps it to another point and *f* maps both points over here (points to one point in C) because *f*'s not 1-1 so that's a counterexample, see what I'm saying."

Pip then drew the correct diagram, as seen in Figure 3. Pip said that this figure does not exclude *f* being onto. At one point Sam said: "*g* could be *x+1* and *h* could be *x+2*" (at this point both B and C had been defined as given), and then they said in unison: "*g* takes 0 to 0 and h takes 0 to 1". Sam pointed to *h* and said "that could just be a translation." After some discussion and explanation for Sam, Sam then said "we could have *f* map everything onto the same point in C". Pip said that then *f* would not be onto, but Sam countered that C could consist of only one point. They proceeded to construct the counterexample where A=*{0}*, B=*{0,1}*, and C=*{0}*, *g(0)=0*, *h(0)=1* and *f(x)=0*.

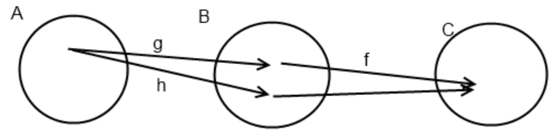

Figure 3: Correct interpretation of the problem.

Commentary on Part 1

The students initially tried to construct an example of a function to get more insight into the problem while almost simultaneously trying to understand the definition of 'onto'. They also used a diagrammatic representation from the start, so have made two moves which in the S/S distinction would be seen as semantic. The diagram is manipulable (M), whereas they were unable to manipulate the example they have chosen. They then tried to develop a 'less complicated' example, using the arrow representation as a technical handle (TH) to illustrate what might be possible (G), and they appeared to get some conceptual insight (CI) about the statement being false. They then developed a counter-example (A). The problem formulation of 'prove or disprove' seemed to have prompted a need for insight into whether the statements were true or false. They had drawn on personal example spaces (PES) of simple functions and of general representations to provide manipulable objects. The representation embedded a CI about possible routes between domains and images more obviously than did their algebraic examples. This suggests that it was helpful that students' PES afforded movement from one representation to another until they gained insight into proving the statement. This could be described as finding an alignment between CI and TH through alighting on an appropriate representation. However, only one student seemed to have this facility. Pip wanted C to be the real numbers, but Sam talked him into using *{0}*. It is plausible that Pip's past experience of functions which have the reals as the domain has led to *R* being treated as a prototypical function domain. We also note here the shift from a general class to finding a single counterexample, and Sam seemed to want formulae for *g* and *h* even though A, B and C could only have 1 or 2 values. It is possible that Sam's PES of functions consists of formulae. The two students appeared to want different levels of abstraction for the final articulation and also used numerical examples differently, possibly because of their different PES. The construction of a special, economical, numerical example in this case to counter a statement and to demonstrate the underlying structure seems to straddle the S/S distinction.

Students' Work on Part 2

When given part 2 of the problem, the students wrote it down with a definition of 'one-to-one'. Pip drew Figure 4 and said: "It's obvious this time that if it is mapping to a point (from B to C) it's only coming from one because that's the one-to-one part." He drew arrow from B to C in the figure and continued "But here's the problem for me. Does the fact it is not 'onto' mean anything?" They convinced themselves that it does not matter. They then drew the two arrows from A to B in Figure 4. Sam then started talking about $g(x)=x+1$ and $h(x)=x+2$, examples proffered but discarded in part 1, and drew Figure 5. Pip was at first convinced that this was a counterexample, but then realized that both *g* and *h* must start from the same point.

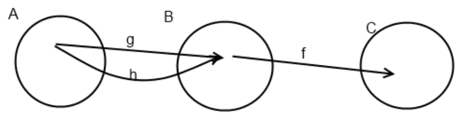

Figure 4: Initial understanding of part 2 of problem.

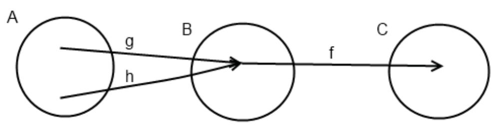

Figure 5: A misunderstanding of problem.

Pip said: "We are trying to show for every x, $f(g(x))=f(h(x))$, but this shows there is some x_1 and x_2 such that $f(g(x_1))=f(h(x_2))$ and these are different." Pip still gets confused and is not sure what this shows, so he says, "let's take your example, $g(0)=2$ and $h(1)=2$.... don't we have to show that $g(0)$ does not equal $h(0)$?... and then you have to show $f(g(0))=f(h(0))$ … (he then started to draw Figure 6) and from your example *h* has to map here (draws *h* arrow from same point as *g* arrow in A to different point in B) but then *f* has to map back to the same point on C (draws arrow from B to C to same point as other arrow) NOT ONE-TO-ONE" and slams pen down. From here, they were able to write a proof of the statement using the diagram; they assumed $g \neq h$ which means for some a, $g(a) \neq h(a)$. Since f is one-to-one, $f(g(a)) \neq f(h(a))$, hence contradiction.)

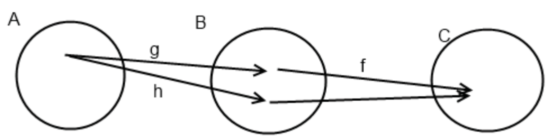

Figure 6: Basis for proof by contradiction.

Commentary on Part 2

We note first that if our earlier conjectures about students' PES for functions and their domains are true, their PES did provide the appropriate tools for a proof. However, the diagrammatic approach that was successful for the students in part 1 did not appear to help the students in part 2 and so Pip resorted to numerical examples. This to-and-fro between numerical examples and diagrams provided the basis for a proof by contradiction. The example was not used to signify generality at first, but to explore structure. In part 2 (similarly to part 1) the students tried to work deductively with definitions of one-to-one and onto, and resorted to the diagrams and examples to 'get a sense of' the definition. This is *relational necessity*: for these students in this situation examples were necessary for them to understand the mathematical relations. They could not work with the symbols until they understood the effect of 'one-to-one'. This does not imply that this would be true for everyone tackling this problem, nor for these students for all problems. The formulation of this problem is such that part 1, which requires a counterexample, may have established exemplification as a useful approach for part 2. They had chosen a representation that allowed the definitions to be drawn and seen as mappings, rather than remain as abstract qualities. Alignment of the CI, that is the nature of 'one-to-one' functions, and the TH, that is the possible mappings from the same input, came about through their use of the particular representation involving arrows and 'blobs' for sets.

CONCLUSION

This discussion gives a view of our analytical approach and also typifies the kind of behaviour we observed. This problem situation had the four required characteristics for the use of examples for proving that we listed in the introduction, i.e., suitable experience of utility of examples; problem formulation; suitable PES; and there was relational necessity. It also illustrates Alcock and Weber's (2010) requirements, i.e., the ability to exemplify; that examples suggest inferences; that examples are 'correct'; and that examples relate to formal definitions. We also need to state that our whole sample of videos included several pairs who began by trying to prove a statement and resorted to examples only if relational necessity arose, thus showing that the 'Introduction to Proof' course had not imposed exemplification as a necessary action in proving.

We are able to say more: that it is the alignment of CI and TH within a suitable choice of representation that appeared to lead to a method of proof in many of our videos as it did above, that is the finding of a technical handle that somehow models the students' emerging grasp of the underlying concepts.

Figure 7 represents the integration of CI & TH with MGA that we have demonstrated above. We would not claim that a syntactic approach might provide a technical handle while a semantic approach provides CI, although it was tempting to assume this when we set out to do our analysis. Rather, in most of our videos either the TH or the CI could arise while translating between representations. Finding a representation that aligns TH and CI was a key step in all the proof processes. This can be seen in Figure 7. In only one of our cases was alignment found in the representation in which the problem was posed.

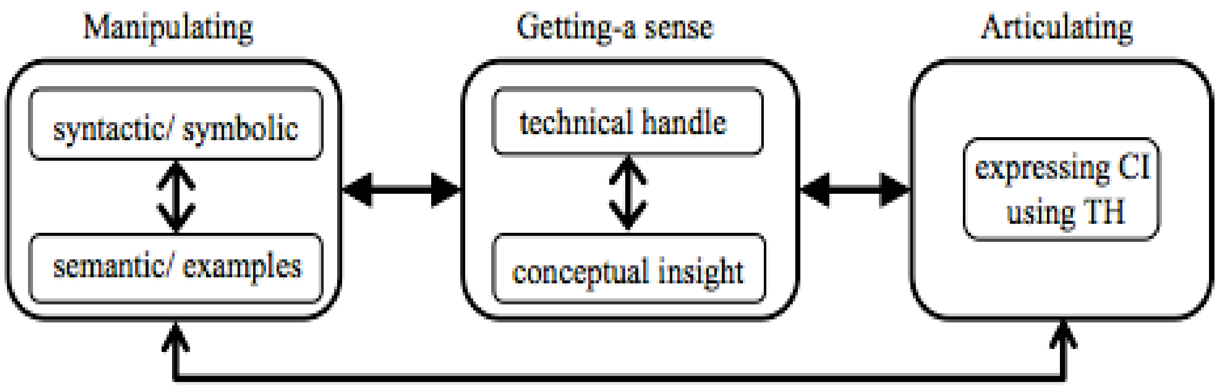

Figure 7: Integration of CI/TH with MGA.

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