Propositional Logic

TUTORIAL

Syntax & Grammar

Sentence → AtomicSentence | ComplexSentence

AtomicSentence → True | False | P | Q | R | . . .

ComplexSentence → (Sentence) | [Sentence]
| ¬Sentence
| Sentence ∧ Sentence
| Sentence ∨ Sentence
| Sentence ⇒ Sentence
| Sentence ⇔ Sentence

OPERATOR PRECEDENCE : ¬, ∧, ∨,⇒,⇔

Truth tables for the five logical connectives

P	Q	¬P	P ∧ Q	P ∨ Q	P ⇒ Q	P ⇔ Q
False	False	True	False	False	True	True
False	True	True	False	True	True	False
True	False	False	False	True	False	False
True	True	False	True	True	True	True

Validity – Satisfiability

- A sentence is **valid** if it is true in *all* models. Valid sentences are also known as **tautologies**—they are *necessarily* true. (e.g. $P \lor \neg P$)
- A sentence is **satisfiable** if it is true in, or satisfied by, *some* model. (e.g. $P \lor \neg Q$)
- A sentence is **unsatisfiable** if it is false in, or unsatisfied by, *any* model. (e.g. $P \land \neg P$)
- There is at least one **model** for a sentence, if there is an interpretation I that satisfies this sentence. I is the model of this sentence.
- Horn clause is a disjunction of literals of which *at most one is positive*. (e.g. $\neg P_1 \lor \neg P_2 \lor Q$)

Validity – Satisfiability Example 1

A	B	C	$A \Rightarrow B$	$A \wedge C$	$(A \wedge C) \Rightarrow B$	$(A \Rightarrow B) \Rightarrow ((A \wedge C) \Rightarrow B)$
F	F	F	T	F	T	T
F	F	T	T	F	T	T
F	T	F	T	F	T	T
F	T	T	T	F	T	T
T	F	F	F	F	T	T
T	F	T	F	T	F	T
T	T	F	T	F	T	T
T	T	T	T	T	T	T

Sentence is valid, satisfiable, a tautology and has a model.

Reasoning – Example 2

Premises

- If John is a student at DI, then he is a nerd.
- If John wakes up early in the morning, then he is not cool.
- If John is a nerd, then he wakes up early in the morning.

Conclusion

If John is a student at DI, then he is not cool.

Conjunctive Normal Form

Every sentence of propositional logic is logically equivalent to a conjunction of clauses

CNF Transformation

- Eliminate ⇔, replacing A ⇔ B ≡ (A ⇒ B) ∧ (B ⇒ A).
- Eliminate ⇒, replacing A ⇒ B ≡ ¬A ∨ B.
- CNF requires ¬ to appear only in literals, so we "move ¬ inwards" by repeated application of the following equivalences:
  - ¬(¬A) ≡ A (double-negation elimination)
  - ¬(A ∧ B) ≡ (¬A ∨ ¬B) (De Morgan)
  - ¬(A ∨ B) ≡ (¬A ∧ ¬B) (De Morgan)
- Distributive property
  - (A ∧ B) ∨ C ≡ (A ∨ C) ∧ (B ∨ C)
  - C ∨ (A ∧ B) ≡ (C ∨ A) ∧ (C ∨ B)

Proof by resolution – Example 2

In order to show that KB |= A, we show that (KB ∧ ¬A) is unsatisfiable. We do this by proving a contradiction.

- Codify natural language premises and conclusion into propositional logic sentences.
  1. JSDI ⇒ JN
  2. JWUE ⇒ ¬JC
  3. JN ⇒ JWUE

  JSDI ⇒ ¬JC

- Transform our premises and conclusion to CNF
  1. ¬JSDI V JN
  2. ¬JSWUE V ¬JC
  3. ¬JN V JWUE

  ¬JSDI V ¬JC

Proof by resolution – Example 2

- Add the negation of conclusion to the KB
  1. ¬JSDI ∨ JN
  2. ¬JSWUE ∨ ¬JC
  3. ¬JN ∨ JWUE
  4. JSDI
  5. JC
- Apply resolution
  6. (1, 4) $\frac{\neg JSDI \lor JN,\ JSDI}{JN}$
  7. (6,3) $\frac{JN,\ \neg JN \lor JWUE}{JWUE}$
  8. (2,5) $\frac{\neg JSWUE \lor \neg JC,\ JC}{\neg JSWUE}$
  9. (7,8) $\frac{JSWUE,\ \neg JSWUE}{\emptyset}$
- We had a contradiction so KB |= (JSDI ⇒ ¬JC)