Orthogonal Range Searching

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Outline

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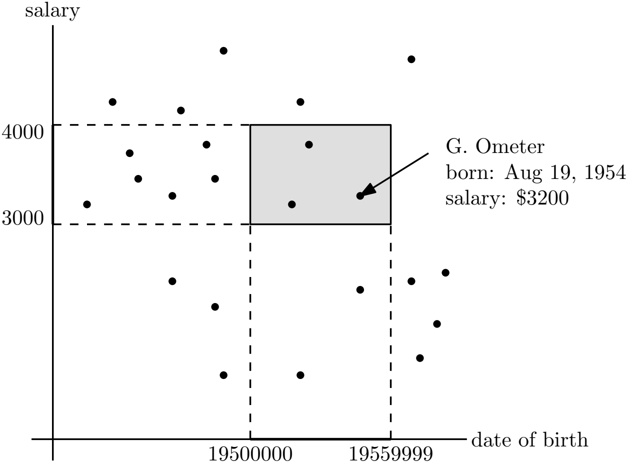
1. Introduction
2. 1-Dimensional Range Searching
3. 2-Dimensional
   - kd-Trees
   - Range Trees
   - Fractional Cascading

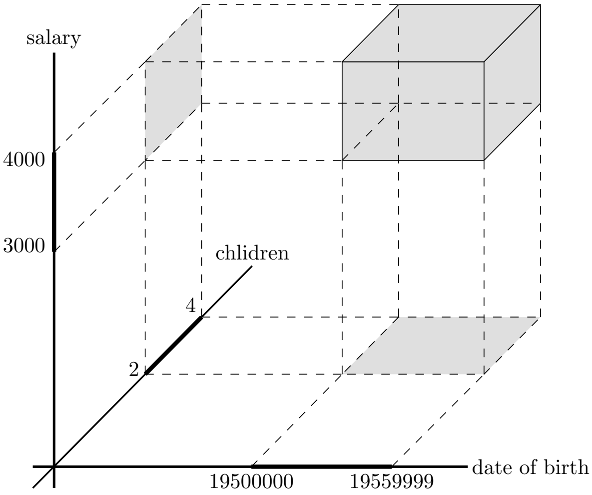
Querying a pointset

- High-dimensional geometry is everywhere: Conformation space, images, the web, bio, . . .
- Queries in a database can be interpreted geometrically: records are high-dim points, record queries are queries on the points.

A typical query interpreted geometrically



A query in 3 dimensions



The geometric approach

- We are interested in answering queries on d fields of the records in our database.
- Transform the records to points in d-dimensional space.
- The transformed query asks for all points inside a d-dimensional axis-parallel box (may be unbounded).
- Such a query is called "rectangular" or "orthogonal" range query.

Lower bounds in $\mathbb{R}^d$

Trivial

$\Omega(k)$, where k = size of output.

Decision tree model

$$S = \{(0, \ldots, 0, x_i, 0, \ldots, 0) : -a \leq x_i \leq a, x_i \neq 0\}, \ a \in \mathbb{N}^*,$$

has $n = 2da$ points. The following queries return distinct sets $\neq \emptyset$:

$$[-b_1, c_1] \times \cdots \times [-b_d, c_d], \ 1 \leq b_i, c_i \leq a.$$

There are a^{2d} queries. A decision tree has log height $= \Omega(d \lg n)$.

Semigroup ops

Dynamic data structures have query time $= \Omega((\lg n)^d)$ [Feldman'81]

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The problem

Data

A set of points $P = \{p_1, p_2, \ldots, p_n\}$ in 1-dimensional space (a set of real numbers).

Query

Which points lie inside a “1-dimensional query rectangle”? i.e. inside an interval $[x : x']$?

Efficient Data Structures

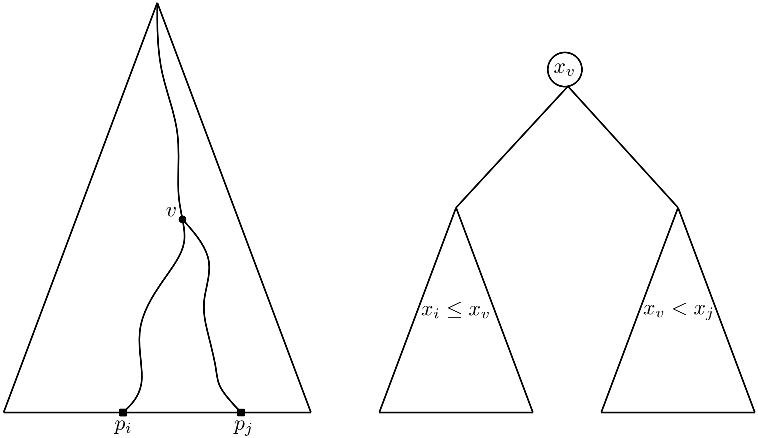
Arrays

- Solve it: $O(n)$ space, $O(n \log n)$ preprocess, $O(k + \log n)$ query
- But, do not generalize in higher dim,
- do not allow efficient updates: $O(n)$.

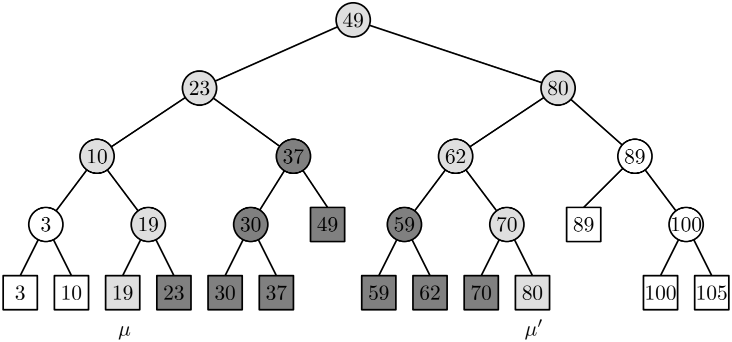
Balanced Binary Search Trees (BBST)

- The leaves of T store the points of P,
- internal nodes store splitting values that guide the search.

Balanced Binary Search Trees



A search with the interval $[18 : 77]$



A search with the interval $[x, x']$

- Search for x and x' in T. The search ends to leaves μ and μ'.
- Report all points stored at leaves between μ and μ' plus, possibly, the points stored at μ and μ'.

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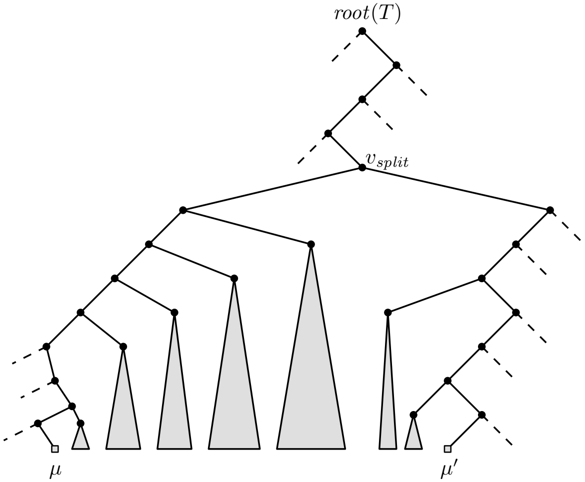
Remark

The leaves to be reported are the ones of subtrees that are rooted at nodes whose parents are on the search paths to μ and μ'.

The selected subtrees



Algorithms

FindSplitNode(T, x, x')

$v \leftarrow root(T)$
while v is not a leaf **and** $(x' \leq x_v$ **or** $x > x_v)$ **do**
  if $x' \leq x_v$ **then**
    $v \leftarrow lc(v)$
  else
    $v \leftarrow rc(v)$
return v

1D-RangeQuery$(T, [x : x'])$

$v_{split} \leftarrow$ FindSplitNode(T, x, x')
if v_{split} is a leaf **then**
  check if $x_{v_{split}}$ must be reported
else {follow the path to x}
  $v \leftarrow lc(v_{split})$
  while v is not a leaf **do**
    if $x \leq x_v$ **then**
      ReportSubtree$(rc(v))$
      {subtrees right of path}
      $v \leftarrow lc(v)$
    else
      $v \leftarrow rc(v)$
  check if x_v must be reported
  ...

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- Any reported point lies in the query range.
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- $\Theta(n)$ worst case case query cost.

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- $\Theta(n)$ worst case case query cost.
- $O(k + \log n)$ output sensitive query cost: $O(k)$ to report the points plus $O(\log n)$ to follow the paths to x, x'.

Priority queue

- Implemented by heap: min element at root.
- $O(n)$ storage.
- $O(n)$ construction time.
- $O(\log n)$ update time.
- $O(n)$ for $[a, a']$ query, $O(k)$ for $(-\infty, a']$

Outline

The problem

Data

A set of points $P = \{p_1, p_2, \ldots, p_n\}$ in the plane.

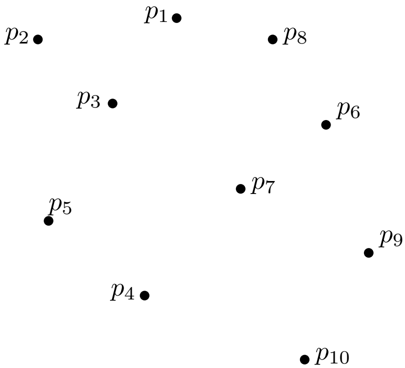
Query

Which points lie inside a query rectangle $[x : x'] \times [y : y']$?

Remark

Point $p = (p_x, p_y)$ lies inside this rectangle iff $p_x \in [x, x']$ and $p_y \in [y, y']$.

The way the plane is subdivided



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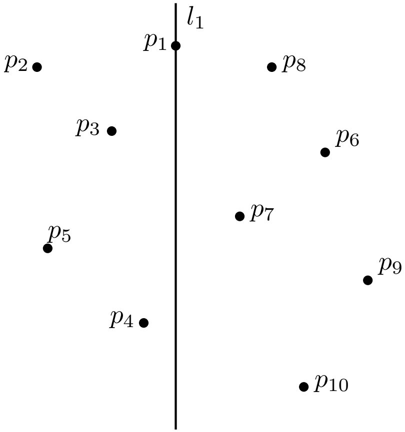

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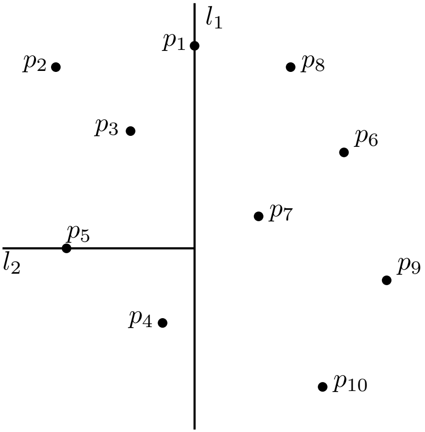

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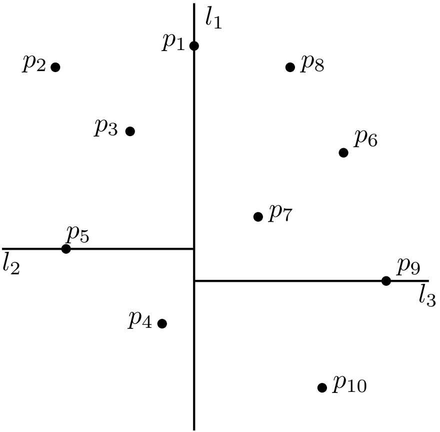

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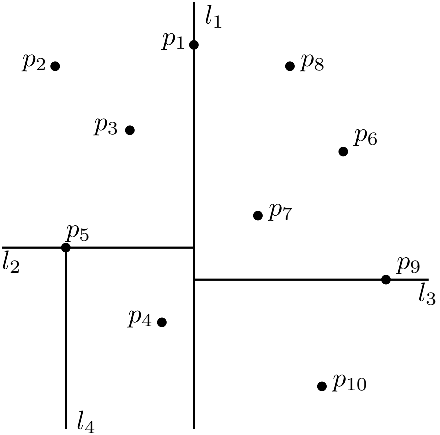

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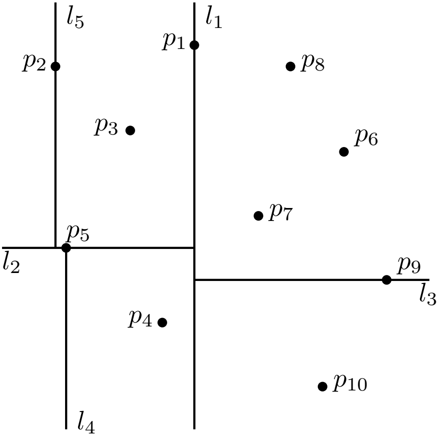

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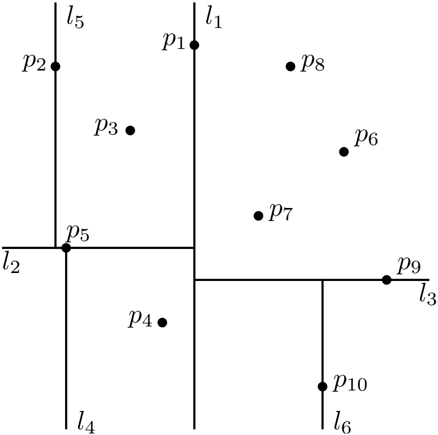

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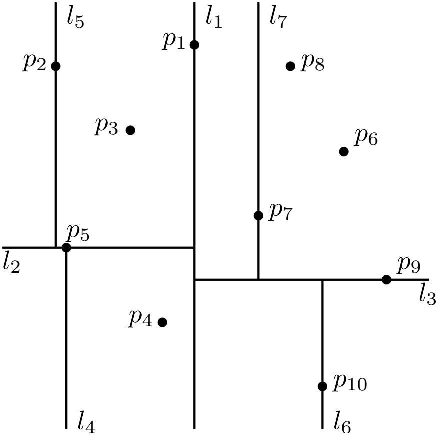

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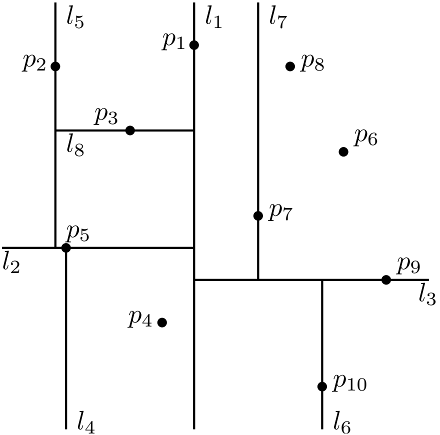

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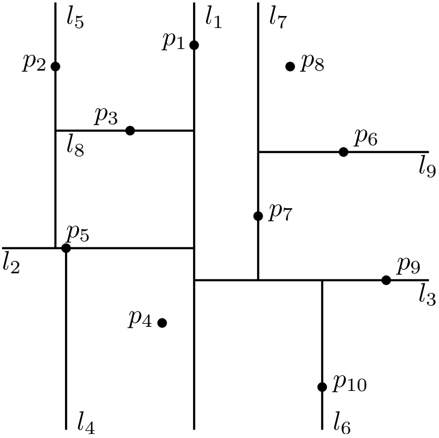
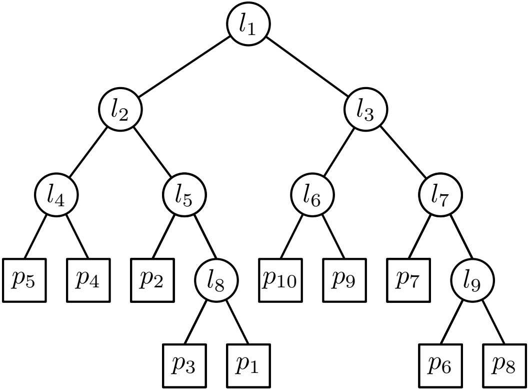


The corresponding binary tree



Algorithm

BuildKdTree$(P, depth)$

if P contains only one point **then**
  return a leaf storing this point
else
  if *depth* is even **then**
    split P with vertical l through median x-coord of points in P
    $P_1 \leftarrow$ the set of points left of l or on l
    $P_2 \leftarrow$ the set of points right of l
  else
    split P with horizontal l through median y-coord of points in P
    $P_1 \leftarrow$ the set of points below l or on l
    $P_2 \leftarrow$ the set of points above l
  $v_{left} \leftarrow \text{BuidKdTree}(P_1, depth + 1)$
  $v_{right} \leftarrow \text{BuidKdTree}(P_2, depth + 1)$
  create a node v storing l
  $lc(v) \rightarrow v_{left}$
  $rc(v) \rightarrow v_{right}$
  return v

Building time and storage

Remarks

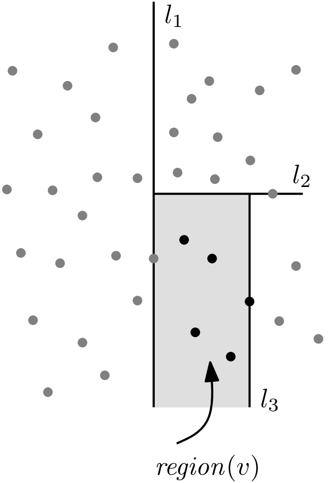
- We split at the $\frac{n}{2}$-th smallest (median) coordinate: $O(n)$ time.
- Preprocessing involves sorting both on x- and y-coordinate.

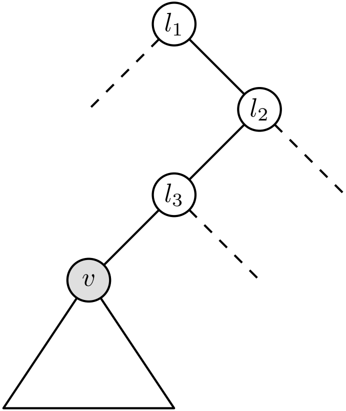
- The building time satisfies the recurrence:

$$T(n) = \begin{cases} O(1) & \text{if } n = 1 \\ O(n) + 2T(\frac{n}{2}) & \text{if } n > 1 \end{cases}$$

- $T(n) = O(n \log n)$ which subsumes the preprocessing time.
- Finding median $= O(n)$
- $O(n)$ storage: points stored at leaves, leaf contains bucket of ≥ 1 points; alternatively stored at internal (splitting) nodes.

Nodes in a kd-tree and regions in the plane

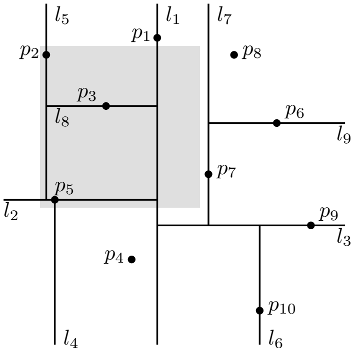


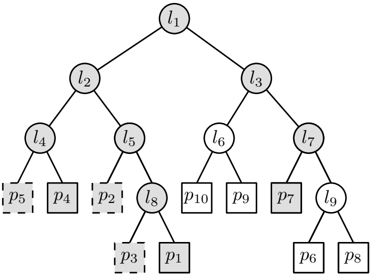


Regions and the query algorithm

- Internal nodes of a kd-tree correspond to rectangular regions of the plane: can be unbounded on one or more sides.
- Regions of nodes at a specific level partition the plane.
- $\mathit{region}(\mathit{root}(T))$ is the whole plane.
- Point stored at (leaf of) subtree rooted at v iff it lies in $\mathit{region}(v)$ (alternative: points stored at internal nodes)
- Search the subtree of v only if the query rectangle intersects $\mathit{region}(v)$.

A query on a kd-tree





Algorithm

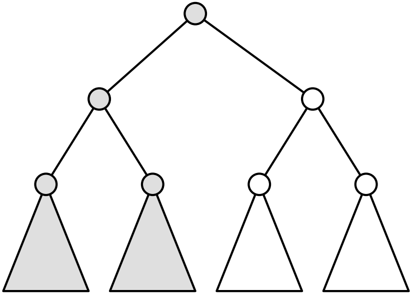
SearchKdTree(v, R)

if v is a leaf **then**
  report point stored at v if in R
else
  if $region(lc(v))$ is fully contained in R **then**
    ReportSubtree($lc(v)$)
  else
    if $region(lc(v))$ intersects R **then**
      SearchKdTree($lc(v), R$)
  if $region(rc(v))$ is fully contained in R **then**
    ReportSubtree($rc(v)$)
  else
    if $region(rc(v))$ intersects R **then**
      SearchKdTree($rc(v), R$)

- Works for any query range R (e.g. triangles).
- $O(k)$ to report k points.
- How many other nodes v are visited?
- For these v, query range intersects $region(v)$

Query time analysis

- Any vertical line intersects $region(lc(root(T)))$ or $region(rc(root(T)))$ but not both.
- If a vertical line intersects $region(lc(root(T)))$ it always intersects the regions corresponding to both children of $lc(root(T))$.



Query time analysis

- The number of intersected regions in a kd-tree storing n points, satisfies the recurrence:

$$Q(n) = \begin{cases} O(1) & \text{if } n = 1 \\ 2 + 2Q(\frac{n}{4}) & \text{if } n > 1 \end{cases}$$

- $Q(n) = O(\sqrt{n})$. The total query time is $O(\sqrt{n} + k)$
- The analysis is rather pessimistic: In many practical situations the query range is small and will intersect much fewer regions.

kd-tree in higher dimensions

kd-trees also used in 3 or a higher dimension d, assuming d not too large; actually interesting for $d \ll \lg n$.

- storage $= O(d \cdot n)$
- construction of balanced tree: $O(d \cdot n \log n)$ by sorting per dimension, $O(n \log n)$ by linear-time median computation.
- insert/delete into balanced kd-tree $= O(\log n)$.
- range query $= O(d \cdot n^{1-1/d} + k)$.
- nearest neighbor $= O(d \cdot n^{1-1/d})$, but $O(\log n)$ expected for sufficiently random (not necessarily uniform) points.

The Range Tree approach

- 2D range queries are two 1D range queries one on $x-$ and one on $y-$coordinate.

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- Find first the points whose $x-$coordinate lies in $[x : x']$ and worry about the $y-$coordinate later.
- During the 1D range query a logarithmic number of subtrees is selected.
- The leaves of these subtrees contain exactly the points whose $x-$coordinate lies in $[x : x']$.

The Range Tree approach

Canonical Subset of a node v

The subset of points $P(v)$ of P stored in the leaves of the subtree rooted at v; clearly, $P(root(\mathcal{T})) = P$.

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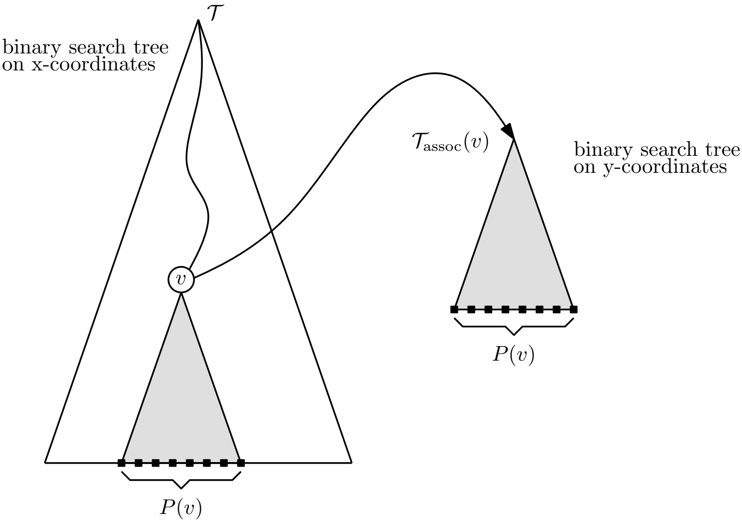
- The subset of points whose $x-$coordinate lies in the query range is a disjoint union of $O(\log n)$ canonical subsets.
- We are not interested in all the points in such subsets.
- Report the ones whose $y-$coordinate lies in $[y : y']$: This is another 1D query.

A 2-dimensional Range Tree



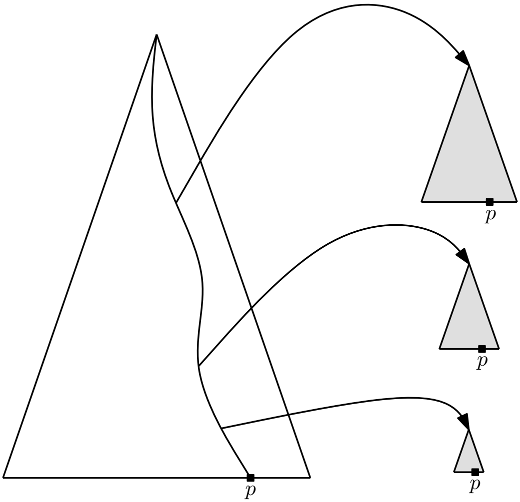
Algorithm

Build2DRangeTree(P)

Build a BST $\mathcal{T}_{assoc}$ on the set P_y
Store the points of P at the leaves of $\mathcal{T}_{assoc}$
if P contains only one point **then**
  Create a leaf v storing this point
  Associate $\mathcal{T}_{assoc}$ with v
else
  Split P into P_{left} and P_{right} through x_{mid}
  $v_{left} \leftarrow$ Build2DRangeTree(P_{left})
  $v_{right} \leftarrow$ Build2DRangeTree(P_{right})
  create a node v storing x_{mid}
  $lc(v) \leftarrow v_{left}$
  $rc(v) \leftarrow v_{right}$
  Associate $\mathcal{T}_{assoc}$ with v
return v

- Preprocess maintains 2 lists of points:
- sorted on $x-$coordinate and on $y-$coordinate.
- Time spent at node in main tree: linear in size of its canonical subset.

Range Tree storage



Storage and costruction time

- Each point is stored only once at a given depth.
- The total depth is $O(\log n)$: the amount of storage is $O(n \log n)$.
- Time spent at node in main tree is linear in size of its canonical subset, hence total construction time equals amount of storage.
- Presorting is $O(n \log n)$.
- Total construction time is $O(n \log n)$.

Algorithm

2DRangeQuery$(\mathcal{T}, [x : x'] \times [y : y'])$

$v_{split} \leftarrow$ FindSplitNode$(\mathcal{T}, x, x')$
if v_{split} is a leaf **then**
  check if $x_{v_{split}}$ must be reported
else {follow the path to x}
  $v \leftarrow lc(v_{split})$
  while v is not a leaf **do**
    if $x \leq x_v$ **then**
      1DRangeQuery$(\mathcal{T}_{assoc}(rc(v)), [y : y'])$
      $v \leftarrow lc(v)$
    else
      $v \leftarrow rc(v)$
  check if x_v must be reported
  ...

Query time analysis

- Time to report points whose $y-$coordinate $\in [y : y']$ is $O(\log n + k_v)$, where $k_v = \#$points reported in this call.

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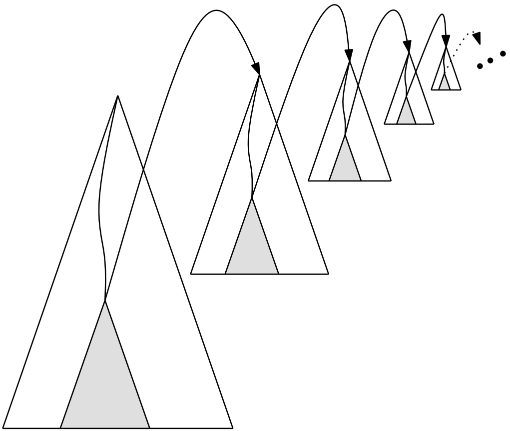
- $Q(n) = O(\log^2 n + k)$.

Higher-Dimensional Range Trees



Higher-Dimensional Range Trees

- P is a set on n points in $d-$dimensional space ($d \geq 2$):
- $O(n \log^{d-1} n)$ storage,
- $O(n \log^{d-1} n)$ construction time,
- $O(\log^d n + k)$ query time.

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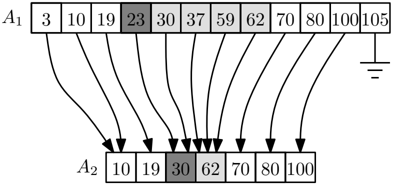
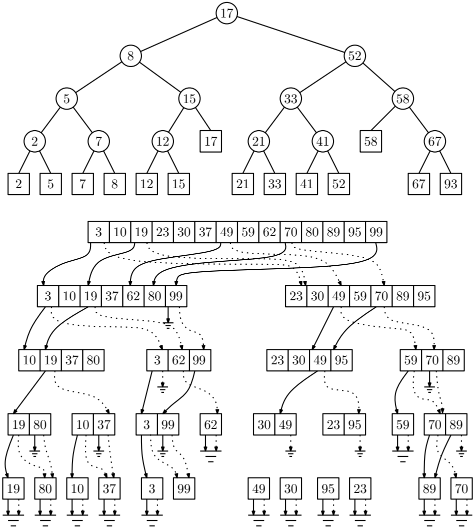


A Layered Range Tree



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- For all $O(\log n)$ nodes on the paths to x, x', maintain pointers to entries in A whose $y-$coordinate $\geq y$, in $O(1)$ time.

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- For all $O(\log n)$ nodes on the paths to x, x', maintain pointers to entries in A whose $y-$coordinate $\geq y$, in $O(1)$ time.
- Report points of $A(v)$ in $O(1 + k_v)$ time, $k_v = \#$reported points at node v.

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- At v_{split} find the entry in $A(v_{split})$ whose $y-$coordinate $\geq y$, in $O(\log n)$ time.
- For all $O(\log n)$ nodes on the paths to x, x', maintain pointers to entries in A whose $y-$coordinate $\geq y$, in $O(1)$ time.
- Report points of $A(v)$ in $O(1 + k_v)$ time, $k_v = \#$reported points at node v.
- Total query time becomes $O(\log n + k)$.

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- [Chazelle-Guibas'86]