ΜΑΘΗΜΑ 23 (02-05-2011)

Υπενθύμιση

$f : B(\subseteq \mathbb{R}^3) \to \mathbb{R}$ συνεχής

B xy-απλό ή yx-απλό ή ..., απλό

$\vec{T} : \mathbb{R}^3 \to \mathbb{R}^3$, $\vec{T}(u,v,w) = \left(x(u,v,w), y(u,v,w), z(u,v,w)\right)$ μετασχηματισμός 1-1

$$J\vec{T}(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)}(u,v,w) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}(u,v,w)$$

με ορίζουσα $\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}(u,v,w)\right| \neq 0$

Τότε
$$\boxed{\iiint_B f(x,y,z)\,dx\,dy\,dz = \iiint_{\vec{T}^{-1}(B)} f\left(x(u,v,w), y(u,v,w), z(u,v,w)\right) \left\| \frac{\partial(x,y,z)}{\partial(u,v,w)}(u,v,w) \right\| du\,dv\,dw}$$

απόλυτος τιμή — ορίζουσα.

▸ $f \equiv 1$:

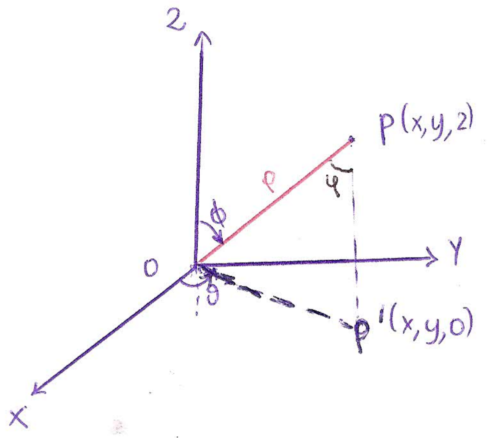
$$\boxed{V(B) = \iiint_{\vec{T}^{-1}(B)} \left\| \frac{\partial(x,y,z)}{\partial(u,v,w)}(u,v,w) \right\| du\,dv\,dw}, \text{ όγκος του } B$$

Ασκήσεις II : (Αλλαγή μεταβλητών σε σφαιρικές)

Σημείωση: Χρησιμοποιούμε σφαιρικές συντεταγμένες όταν στο πρόβλημά μας υπάρχει συμμετρία ως προς ΣΗΜΕΙΟ!

Υπενθύμιση: Σφαιρικές συντεταγμένες

$\varphi = \sqrt{x^2+y^2+z^2}$



$\theta = \measuredangle(Ox, OP)$

$\varphi = \measuredangle(Oz, OP)$

$x = OP'\sigma\upsilon\nu\theta = \rho\cdot\eta\mu\phi\cdot\sigma\upsilon\nu\theta$

$y = OP'\eta\mu\theta = \rho\cdot\eta\mu\phi\cdot\eta\mu\theta$

$z = \rho\cdot\sigma\upsilon\nu\phi = \rho\cdot\sigma\upsilon\nu\phi$

Ορίζουσα: Σφ. Μετασχηματισμού

$\vec{r}(\rho,\theta,\phi) = (\rho\cdot\sigma\upsilon\nu\theta\cdot\eta\mu\phi,\ \rho\cdot\eta\mu\theta\cdot\eta\mu\phi,\ \rho\cdot\sigma\upsilon\nu\phi),\ \rho>0,\ \theta\in[0,2\pi],\ \varphi\in[0,\pi]$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{pmatrix} \sigma\upsilon\nu\theta\eta\mu\phi & -\rho\,\eta\mu\theta\cdot\eta\mu\phi & \rho\,\sigma\upsilon\nu\theta\sigma\upsilon\nu\phi \\ \eta\mu\theta\eta\mu\phi & \rho\cdot\sigma\upsilon\nu\theta\cdot\eta\mu\phi & \rho\cdot\eta\mu\theta\sigma\upsilon\nu\phi \\ \sigma\upsilon\nu\phi & 0 & -\rho\cdot\eta\mu\phi \end{pmatrix}$$

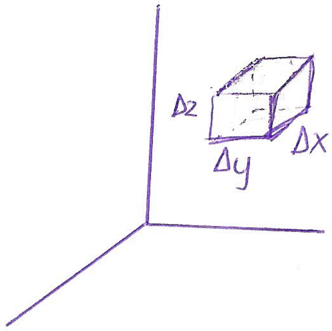
ορίζουσα τα $\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \sigma\upsilon\nu\varphi\left(-\rho^2\eta\mu^2\theta\eta\mu\phi\sigma\upsilon\nu\phi - \rho^2\sigma\upsilon\nu^2\theta\eta\mu\phi\sigma\upsilon\nu\phi\right) - \rho\eta\mu\phi\left(\rho\cdot\sigma\upsilon\nu^2\theta\eta\mu^2\phi + \rho\cdot\eta\mu^2\theta\cdot\eta\mu^2\phi\right)$

$= -\rho^2\sigma\upsilon\nu^2\phi\cdot\eta\mu\phi - \rho^2\cdot\eta\mu\phi\cdot\eta\mu^2\phi = -\rho^2\cdot\eta\mu\phi$

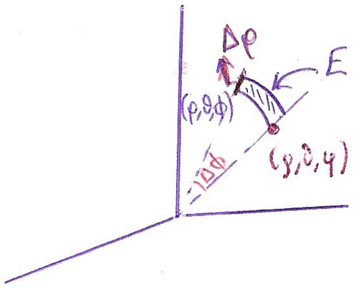
άρα $\left|\left|\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\varphi)}\right|\right| = \rho^2\eta\mu\phi$

$\longrightarrow$

Μια κάποια προσέγγιση του τύπου αλλαγής μεταβλητής

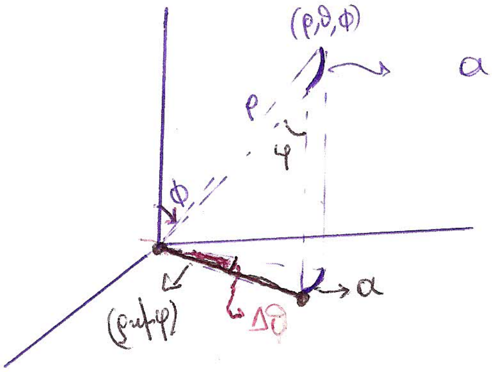


Στις καρτεσιανές συντεταγμένες ο στοιχειώδης όγκος είναι $\Delta x \Delta y \Delta z$. ($\Delta x, \Delta y, \Delta z$ σταθερά). Δεν εξαρτάται από τις κορυφές του ορθογωνίου.



~~θ=σταθερό~~

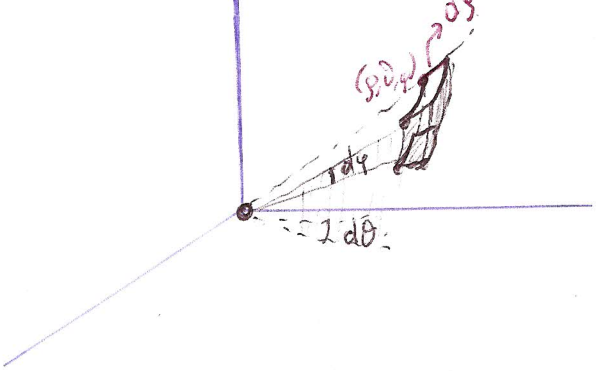
$E = (\rho \cdot \Delta\phi) \cdot \Delta\rho$ (E=εμβαδόν)



$\alpha = (\rho \cdot \eta\mu\phi) \cdot \Delta\theta$

Στοιχειώδης όγκος είναι $E \cdot \alpha = \rho^2 \cdot \eta\mu\phi \cdot \Delta\rho \cdot \Delta\theta \cdot \Delta\phi$

$\Delta x \Delta y \Delta z = \rho^2 \cdot \eta\mu\phi\, \Delta\rho \cdot \Delta\theta \cdot \Delta\phi$



www.tf.uni-kiel.de/matwis/amat/elmat_en/kap_3/basics/b3_2_2.html

Ασκήσεις (συμμετρία ως προς σημείο)*

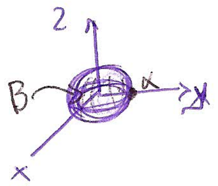
$$\iiint_B f(x,y,z)\,dxdydz = \iiint_{\vec{T}^{-1}(B)} f(\rho\cdot\text{συν}\theta\,\text{ημ}\phi,\ \rho\cdot\text{ημ}\theta\cdot\text{ημ}\phi,\ \rho\cdot\text{συν}\phi)\,\rho^2\cdot\text{ημ}\phi\, d\rho\, d\theta\, d\phi$$

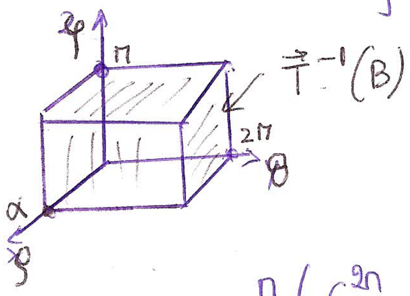
$$V(B) = \iiint_{T^{-1}(B)} (\rho^2\,\text{ημ}\phi)\, d\rho\, d\theta\, d\phi$$

1) Να βρεθεί ο όγκος της σφαίρας

$B = \{(x,y,z)\in\mathbb{R}^3,\ x^2+y^2+z^2\le a^2\}$ $\quad a>0$

$\vec{T}^{-1}(B) = \{(\rho,\theta,\phi) : 0\le\rho\le a,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\pi\}$





$$V(B) = \int_0^{\pi}\left[\int_0^{2\pi}\left(\int_0^{a}\rho^2\,\text{ημ}\phi\, d\rho\right)d\theta\right]d\phi = \int_0^{\pi}\left(\int_0^{2\pi}\frac{a^3}{3}\text{ημ}\phi\, d\theta\right)d\phi =$$

$$= \frac{2\pi a^3}{3}\left(\int_0^{\pi}\text{ημ}\phi\, d\phi\right) = \frac{2\pi a^3}{3}\left(\text{συν}\phi\Big|_0^{\pi}\right) = \frac{2\pi a^3}{3}(1+1) = \frac{4\pi}{3}a^3$$

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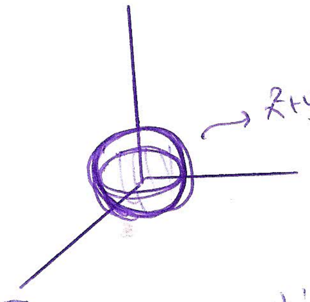
* Μπορούν να λυθούν και με κυλινδρικό μετασχηματισμό

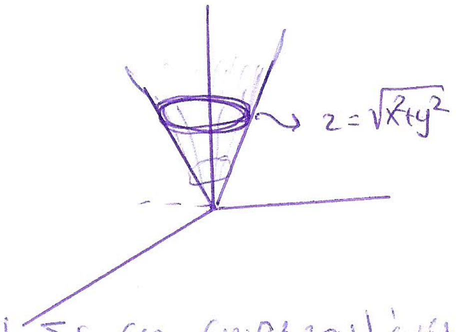
2) $B=\{(x,y,z): x^2+y^2\le z^2,\ x^2+y^2+z^2\le a^2,\ z\ge 0\}$ $(a>0)$

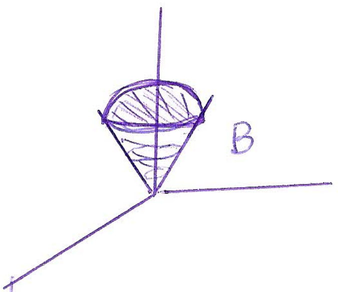
i) Να υπολογιστεί ο όγκος $V(B)$

ii) » το $\iiint_{B'} \sqrt{x^2+y^2+z^2}\,dxdydz$, B' το κομμάτι του B στην 1η στερεά γωνία $(x,y,z\ge 0)$

i)







Σε σφ. συντεταγμένες: $\rho=a$ | Σε σφ. συντεταγμένες $\varphi=\frac{\pi}{4}$

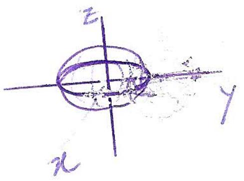
$$\vec{T}^{-1}(B)=\{(\rho,\theta,\phi): 0\le\rho\le a,\ 0\le\theta\le 2\pi,\ 0\le\phi\le\pi/4\}$$

$$V(B)=\int_0^{\pi/4}\int_0^{2\pi}\left(\int_0^a(\rho^2\,\eta\mu\phi)\right)d\theta\, d\phi=\frac{2\pi}{3}a^3\left(1-\frac{\sqrt{2}}{2}\right)$$

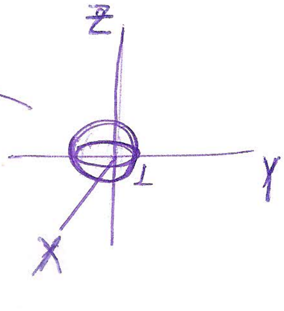
ii) $$I=\int_0^{\pi/4}\int_0^{\pi/2}\int_0^{a}\rho\,(\rho^2\,\eta\mu\phi)\,d\rho\, d\theta\, d\varphi=\pi/8\cdot a^4\left(1-\frac{\sqrt{2}}{2}\right)$$

3) $B=\left\{(x,y,z): \left(\frac{x}{a}\right)^2+\left(\frac{y}{\beta}\right)^2+\left(\frac{z}{\gamma}\right)^2\le 1\right\}$

να υπολογιστεί ο όγκος $V(B)$



(συμμετρία ως προς το σημείο $\xleftarrow{\vec{T}_1}$



$$\vec{T}_1(X,Y,Z)=(aX,\ bY,\ \gamma Z)$$

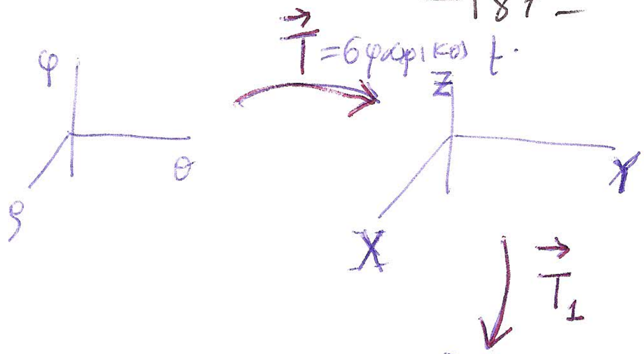
$$\vec{T}_1^{-1}(B)=\{(X,Y,Z): X^2+Y^2+Z^2\le 1\}$$

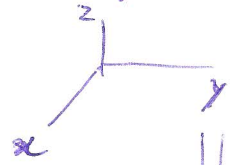
$$x = \frac{X}{\alpha} \quad\Big|\quad x = \alpha \cdot X$$

$$y = \frac{Y}{\beta} \quad\Big|\quad y = \beta \cdot Y$$

$$z = \frac{Z}{\gamma} \quad\Big|\quad z = \gamma \cdot Z$$

$\vec{T}$ = σφαιρικοί t.





Τελικά

i) $X = \rho \cdot \sigma\upsilon\nu\theta \cdot \eta\mu\phi$

$Y = \rho \cdot \eta\mu\theta \cdot \eta\mu\phi$

$Z = \rho \cdot \sigma\upsilon\nu\phi$

$$(\rho,\theta,\varphi) \xrightarrow{\vec{T}} (\rho\,\sigma\upsilon\nu\theta\,\eta\mu\varphi,\ \rho\,\eta\mu\theta\,\eta\mu\varphi,\ \rho\,\sigma\upsilon\nu\varphi)$$

$$\downarrow \vec{T}_1$$

$$\xrightarrow{\vec{T}_1 \circ \vec{T}} (\alpha\rho\,\sigma\upsilon\nu\theta\,\eta\mu\varphi,\ \beta\rho\,\eta\mu\theta\,\eta\mu\varphi,\ \gamma\rho\,\sigma\upsilon\nu\varphi).$$

ορίζουσα $\alpha\beta\gamma\rho^2 \cdot \eta\mu\phi$ του $\vec{T}_1 \circ \vec{T}$

$x = \alpha \cdot \rho \cdot \sigma\upsilon\nu\theta \cdot \eta\mu\phi$

$y = \beta \cdot \rho \cdot \eta\mu\theta \cdot \eta\mu\phi$

$z = \gamma \cdot \rho \cdot \sigma\upsilon\nu\phi$

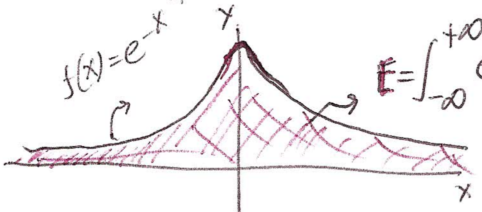
$$V(B) = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} \alpha\beta\gamma\, \rho^2 \cdot \eta\mu\phi \cdot d\rho\, d\theta\, d\phi = \frac{4\pi}{3}\alpha\beta\gamma$$

Άσκηση (http://en.wikipedia.org/wiki/Gaussian_integral)

Να υπολογιστούν τα ολοκληρώματα

i) $$\iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dxdy =: \lim_{\alpha \to +\infty} \iint_{\{(x,y)\,:\,\sqrt{x^2+y^2} \le \alpha\}} e^{-(x^2+y^2)}\,dxdy = \pi$$

ii) ν.δ.ο. $$\int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}\ ,\quad \int_0^{+\infty} e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$$

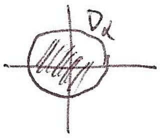


$$E = \int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$$

Ολοκλήρωμα Gauss

i) $\iint e^{-(x^2+y^2)}\,dx\,dy = \int_0^{2\pi}\left(\int_0^{a} r\cdot e^{-r^2}\,dr\right)d\theta = 2\pi\left(-\frac{e^{-r^2}}{2}\Big|_0^{a}\right) = \pi\left(1-e^{-a^2}\right)$

$D_\alpha = \{(x,y) : \sqrt{x^2+y^2} \le a\}$



Άρα $\iint_{\mathbb{R}^2} (e^{-(x^2+y^2)}\,dx\,dy = \pi$ $\quad \left(\lim_{\alpha\to+\infty}(e^{-\alpha^2}) = 0\right)$

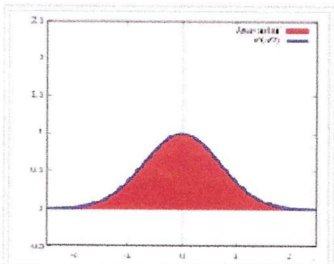
ii) $\pi = \iint_{\mathbb{R}^2} e^{-(x^2+y^2)}\,dx\,dy = \int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)e^{-y^2}\,dy$

$= \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)\cdot\left(\int_{-\infty}^{+\infty} e^{-y^2}\,dy\right) = \left(\int_{-\infty}^{+\infty} e^{-x^2}\,dx\right)^2 \Rightarrow \int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$

Σημ. Το ανωτέρω ολοκλήρωμα σχετίζεται με την Κατανομή του Gauss (ή Κανονική Κατανομή) και ιστορικά ονομάζεται ο "Νόμος των σφαλμάτων". Έχει κεντρικό ρόλο στην Θεωρία Πιθανοτήτων, λόγω του Κεντρικού Οριακού Θεωρήματος. και όχι μόνον.

Gaussian integral

From Wikipedia, the free encyclopedia



A graph of $f(x) = e^{-x^2}$ and the area between the function and the x-axis, which is equal to $\sqrt{\pi}$.

The **Gaussian integral**, also known as the **Euler-Poisson integral** or **Poisson integral**,[1] is the integral of the Gaussian function e^{-x^2} over the entire real line. It is named after the German mathematician and physicist Carl Friedrich Gauss. The integral is:

$$\boxed{\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.}$$

This integral has wide applications. When normalized so that its value is 1, it is the density function of the normal distribution. It is closely related to the error function, which is the same integral with finite limits.

Although no elementary function exists for the error function, as can be proven by the Risch algorithm, the Gaussian integral can be solved analytically through the tools of calculus. That is, there is no elementary *indefinite integral* for $\int e^{-x^2}\,dx$, but the definite integral $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ can be evaluated.

The Gaussian integral is encountered very often in physics and numerous generalizations of the integral are encountered in quantum field theory.

Contents

Computation

By polar coordinates ('Εγινε στο μάθημα)

A standard way to compute the Gaussian integral is

- consider the function $e^{-(x^2+y^2)} = e^{-r^2}$ on the plane $\mathbf{R}^2$, and compute its integral two ways:
- on the one hand, by double integration in the Cartesian coordinate system, its integral is a square:

- on the other hand, by shell integration (a case of double integration in polar coordinates), its integral is computed to be π.

Comparing these two computations yields the integral, though one should take care about the improper integrals involved.

Brief proof

Briefly, using the method above, one computes that on the one hand,

On the other hand,

$$\begin{aligned}\int_{\mathbf{R}^2} e^{-(x^2+y^2)}\,dA &= \int_0^{2\pi}\int_0^{\infty} e^{-r^2} r\,dr\,d\theta \\ &= 2\pi \int_0^{\infty} re^{-r^2}\,dr \\ &= 2\pi \int_{-\infty}^{0} \frac{1}{2}e^{s}\,ds = \pi \int_{-\infty}^{0} e^{s}\,ds = \pi(e^0 - e^{-\infty}) \\ &= \pi(1-0) = \pi,\end{aligned}$$

where the factor of r comes from the transform to polar coordinates ($r\,dr\,d\theta$ is the standard measure on the plane, expressed in polar coordinates), and the substitution involves taking $s = -r^2$, so $ds = -2r\,dr$.

Combining these yields

$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi,$$

so

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

Careful proof

To justify the improper double integrals and equating the two expressions, we begin with an approximating function:

$$I(a) = \int_{-a}^{a} e^{-x^2}\,dx.$$

so that the integral may be found by

$$\lim_{a\to\infty} I(a) = \int_{-\infty}^{\infty} e^{-x^2}\,dx,$$

since

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx < \int_{-\infty}^{-1} -xe^{-x^2}\,dx + \int_{-1}^{1} e^{-x^2}\,dx + \int_{1}^{\infty} xe^{-x^2}\,dx < \infty.$$

Taking the square of *I(a)* yields

$$\begin{aligned} I(a)^2 &= \left(\int_{-a}^{a} e^{-x^2}\,dx\right)\cdot\left(\int_{-a}^{a} e^{-y^2}\,dy\right) \\ &= \int_{-a}^{a}\left(\int_{-a}^{a} e^{-y^2}\,dy\right) e^{-x^2}\,dx \\ &= \int_{-a}^{a}\int_{-a}^{a} e^{-(x^2+y^2)}\,dx\,dy.\end{aligned}$$

Using Fubini's theorem, the above double integral can be seen as an area integral

taken over a square with vertices $\{(-a, a), (a, a), (a, -a), (-a, -a)\}$ on the *xy*-plane.

Since the exponential function is greater than 0 for all real numbers, it then follows that the integral taken over the square's incircle must be less than $I(a)^2$, and similarly the integral taken over the square's circumcircle must be greater than $I(a)^2$. The integrals over the two disks can easily be computed by switching from cartesian coordinates to polar coordinates:

(See to polar coordinates from Cartesian coordinates for help with polar transformation.)

Integrating,

By the squeeze theorem, this gives the Gaussian integral

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

By Cartesian coordinates

Georgakis[2] wrote that the following is "a better alternative to the usual method of reduction to polar coordinates".
(1994)

Let

$$y = xs$$
$$dy - x\,ds.$$

Since the limits on s as y goes to $\pm\infty$ depend on the sign of x, it simplifies the calculation to use the fact that e^{-x^2} is an even function, and, therefore, the integral over all real numbers is just twice the integral from zero to infinity. That is, $\int_{-\infty}^{\infty} e^{-x^2}\,dx = 2\int_0^{\infty} e^{-x^2}\,dx$. Thus, over the range of integration, $x \geq 0$, and the variables y and s have the same limits. This yields:

$$I^2 = 4\int_0^{\infty}\int_0^{\infty} e^{-(x^2+y^2)}dy\,dx.$$

Then

$$\begin{aligned}\frac{I^2}{4} &= \int_0^{\infty}\left(\int_0^{\infty} e^{-(x^2+y^2)}\,dy\right)dx = \int_0^{\infty}\left(\int_0^{\infty} e^{-x^2(1-s^2)}x\,ds\right)dx \\ &= \int_0^{\infty}\left(\int_0^{\infty} e^{-x^2(1+s^2)}x\,dx\right)ds \\ &= \int_0^{\infty}\left[\frac{1}{-2(1+s^2)}e^{-x^2(1-s^2)}\right]_0^{\infty} ds - \frac{1}{2}\int_0^{\infty}\frac{ds}{1+s^2} \\ &= \frac{1}{2}\arctan s\Big|_0^{\infty} = \frac{\pi}{4}.\end{aligned}$$

Finally, $I - \sqrt{\pi}$, as expected.

Relation to the gamma function

The integrand is an even function,

$$\int_{-\infty}^{\infty} e^{-x^2}dx = 2\int_0^{\infty} e^{-x^2}dx$$

Thus, after the change of variable $x = \sqrt{t}$, this turns into the Euler integral

$$2\int_0^{\infty} e^{-x^2}dx = 2\int_0^{\infty}\frac{1}{2}e^{-t}\,t^{-1/2}dt = \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

where Γ is the gamma function. This shows why the factorial of a half-integer is a rational multiple of . More generally,