# Ανάλυση ΙΙ - μάθημα 13 - 23/03/2011

Υπενθύμιση: $f : A(\subseteq \mathbb{R}^n) \to \mathbb{R}$, $\vec{p} \in A$

- Εάν $\exists \ \frac{\partial f}{\partial x_i}$ στο $A$ συνεχείς στο $\vec{p}$

τότε $\exists \ df(\vec{p}) : \mathbb{R}^n \to \mathbb{R}$ Γραμμική

$$df(\vec{p})(\vec{h}) = \nabla f(\vec{p}) \cdot \vec{h} = h_1 \frac{\partial f}{\partial x_1}(\vec{p}) + \dots + h_n \frac{\partial f}{x_n}(\vec{p})$$

- $\vec{f} = (f_1, \dots, f_n) : A \subseteq \mathbb{R}^n \to \mathbb{R}^m$, $f_i : A \to \mathbb{R}$ $i = 1, 2, \dots n$.

$$d\vec{f}(\vec{p}) = (df_1(\vec{p}), \dots, df_n(\vec{p}))$$

$$d\vec{f}(\vec{p})(\vec{h}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}_{(\vec{p})} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix}$$

(συνέχεια) 5) $\vec{f}(x, y) = (x^2 + y^4, \text{συν} x^3, \log(y^2 + 5))$, $d\vec{f}(1, 8)$, $J_{\vec{f}}(1, 8)$

- $f_1(x, y) = x^2 + y^4$ $\begin{cases} \frac{\partial f_1}{\partial x}(x, y) = 2x \\ \frac{\partial f_1}{\partial y}(x, y) = 4y^3 \end{cases}$ συνεχείς στο $\mathbb{R}^2 \Rightarrow$

$$\Rightarrow \exists \ df_1(1, 8)(h_1, h_2) = h_1 \frac{\partial f_1}{\partial x}(1, 8) + h_2 \frac{\partial f_1}{\partial y}(1, 8) =$$

$$= h_1 \cdot 2 + h_2 \, 4 \cdot 8^3 = 2h_1 + 4 \cdot 8^3 h_2$$

- $f_2(x,y) = \sigma\upsilon\nu(x^3)$ $\left\{\begin{array}{l} \dfrac{\partial f_2}{\partial x}(x,y) = -3x^2 \eta\mu x^3 \\ \dfrac{\partial f_2}{\partial y}(x,y) = 0 \end{array}\right.$ Συνεχείς

$$\Longrightarrow \exists\ df_2(1,8)(h_1,h_2) = h_1 \frac{\partial f_2}{\partial x}(1,8) + h_2 \frac{\partial f_2}{\partial y}(1,8)$$

$$= h_1(-3\eta\mu 1) + 0h_2$$

- $f_3(x,y) = \log(\cancel{x}y^2+5)$ $\left\{\begin{array}{l} \dfrac{\partial f_3}{\partial x}(x,y) = 0 \\ \dfrac{\partial f_3}{\partial y}(x,y) = \dfrac{2y}{y^2+5} \end{array}\right.$ Συνεχείς $\Rightarrow$

$$\Longrightarrow \exists\ df_3(1,8)(h_1,h_2) = h_1 \frac{\partial f_3}{\partial x}(1,8) + h_2 \frac{\partial f_3}{\partial y}(1,8) =$$

$$= 0\cdot h_1 + \frac{16}{69} h_2 \quad , (h_1,h_2)\in\mathbb{R}^2$$

Τελικά: $d\vec{f}(1,8)(h_1,h_2) = (2h_1 + 4\cdot 8^3 h_2,\ h_1(-3\eta\mu 1) + 0h_2,\ 0h_1 + \frac{16}{69}h_2)$

$$= \begin{pmatrix} 2 & 4\cdot 8^3 \\ -3\eta\mu 1 & 0 \\ 0 & \frac{16}{69} \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}, \quad J_f(1,8) = \begin{pmatrix} 2 & 4\cdot 8^3 \\ -3\eta\mu 1 & 0 \\ 0 & \frac{16}{69} \end{pmatrix}$$

$(h_1,h_2)\in\mathbb{R}^2.$

6) $\vec{f}(x,y,z)=(x+\text{τοξεφ}\,z^2, e^{xy^2})$,
$d\vec{f}(5,10,1)$, $J_{\vec{f}}(5,10,1)$

• $f_1(x,y,z)=x+\text{τοξεφ}\,z^2$

$$\left.\begin{aligned}\frac{\partial f_1}{\partial x}(x,y,z)&=1\\ \frac{\partial f_1}{\partial y}(x,y,z)&=0\\ \frac{\partial f_1}{\partial z}(x,y,z)&=\frac{2z}{1+z^4}\end{aligned}\right\}$$

Συνεχείς στον $\mathbb{R}^3 \Rightarrow$

$\exists\, df_1(5,10,1)(h_1,h_2,h_3)$
$=h_1+0h_2+h_3\cdot 1$

• $f_2(x,y,z)=e^{xy^2}$

$$\left.\begin{aligned}\frac{\partial f_2}{\partial x}(x,y,z)&=y^2e^{xy^2}\\ \frac{\partial f_2}{\partial y}(x,y,z)&=2xye^{xy^2}\\ \frac{\partial f_2}{\partial z}(x,y,z)&=0\end{aligned}\right\}$$

Συνεχείς στον $\mathbb{R}^3$
$df_2(5,10,1)(h_1,h_2,h_3)=$
$=h_1 10^2 e^{5\cdot 10^2}+h_2 5\cdot 10 e^{5\cdot 10^2}+0h_3$

Τελικά: $d\vec{f}(5,10,1)(h_1,h_2,h_3)=\begin{pmatrix}1 & 0 & 1\\ 100e^{500} & 50e^{500} & 0\end{pmatrix}\begin{pmatrix}h_1\\ h_2\\ h_3\end{pmatrix}$

$J_{\vec{f}}(5,10,1)=\begin{pmatrix}1 & 0 & 1\\ 100e^{500} & 50e^{500} & 0\end{pmatrix}$

7) Να βρεθεί η γραμμικοποίηση των:

i) $f(x) = x^3 + 1$ , $x_0 = 2$

ii) $g(x,y) = xy + \sqrt{x^2+1}$ , $(x_0, y_0) = (5, 1)$

$$f(\vec{x}_0 + \vec{h}) = \underbrace{f(\vec{x}_0) + df(\vec{x}_0)(\vec{h})}_{L(\vec{x}_0+\vec{h})} + \|\vec{h}\|\, g(\vec{h})$$

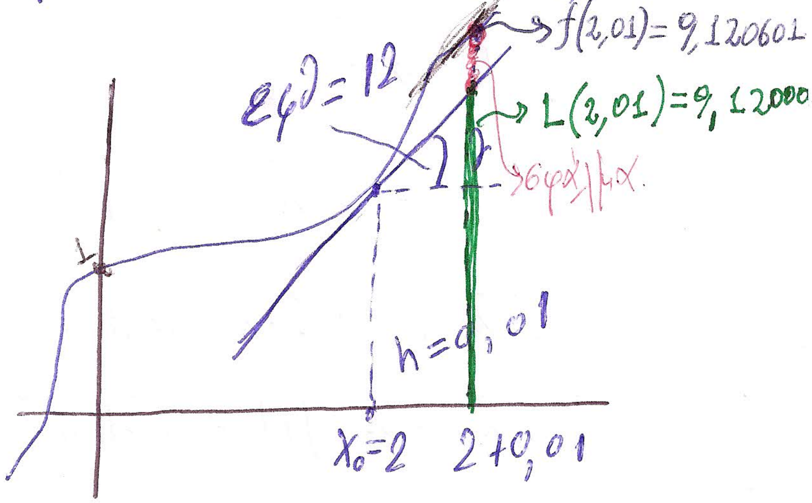
(Το πολυώνυμο 1ου βαθμού που "προσεγγίζει" το $f(\vec{x}_0+\vec{h})$)

(i) $df(2)(h) = 12h$

$L(2+h) = 9 + 12h$

$f(2+h) \simeq 9 + 12h$

π.χ. $\left((2,01)^3 + 1 \simeq 9 + 12(0,01)\right)$




ii) $\dfrac{\partial g}{\partial x}(x,y) = y + \dfrac{x}{\sqrt{x^2+1}}$

$\dfrac{\partial g}{\partial y}(x,y) = x$

συνεχείς στο $\mathbb{R}^2 \Rightarrow$

$\Rightarrow \exists\, dg(5,1)(h_1, h_2)$

$= h_1\left(1 + \dfrac{5}{\sqrt{26}}\right) + 5h_2$

$$L(5+h_1, 1+h_2) = (5 + \sqrt{26}) + \left(1 + \frac{5}{\sqrt{26}}\right) h_1 + 5h_2$$

$$\left( (5+h_1)(1+h_2) + \sqrt{(5+h_1)^2 + 1} \cong (5 + \sqrt{26}) + \left(1 + \frac{5}{\sqrt{26}}\right) h_1 + 5h_2 \right)$$

## Σχέση διαφορισιμότητας - συνέχειας - μερικής παραγώγισης.

$f: A (\subseteq \mathbb{R}^n) \to \mathbb{R}$, $\vec{p} \in A$

$\xrightarrow{***}$ Εάν $\exists\, df(\vec{p}) \Rightarrow f$ είναι συνεχής στο $\vec{p}$

$\not\Leftarrow$ π.χ. $f(x) = |x|$ συνεχής, όχι διαφ. στο $p=0$

$$f(\vec{p}+\vec{h}) = f(\vec{p}) + df(\vec{p})(\vec{h}) + \|\vec{h}\| g(\vec{h}), \quad \lim_{\vec{h}\to\vec{0}} g(\vec{h}) = 0 = g(\vec{0}) \quad (*)$$

$u: \mathbb{R}^n \to \mathbb{R}$ Γραμμική

$u(\vec{x}) = \vec{a}\cdot\vec{x}$, $\vec{x} \in \mathbb{R}^n$, $\vec{a} = (u(\vec{e}_1), \dots, u(\vec{e}_n))$

$|u(\vec{x})| = |\vec{a}\cdot\vec{x}| \leq \|\vec{a}\| \, \|\vec{x}\|$

$|u(\vec{x}) - u(\vec{y})| \leq \|\vec{a}\| \, \|\vec{x} - \vec{y}\|$

$|df(\vec{p})(\vec{h})| = |\nabla f(\vec{p})\cdot\vec{h}| \leq \|\nabla f(\vec{p})\| \, \|\vec{h}\|$

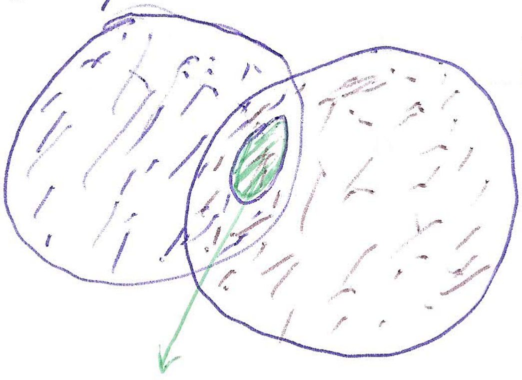
$\lim_{\vec{h}\to\vec{0}} df(\vec{p})(\vec{h}) = 0$

$\overset{(*)}{\Rightarrow} \lim_{\vec{h}\to\vec{0}} f(\vec{p}+\vec{h}) = f(\vec{p})$ δηλ. η $f =$ συνεχής στο $\vec{p}$

$\xrightarrow{***}$ Εάν $\exists\, df(\vec{p}) \Rightarrow \exists\, \frac{\partial f}{\partial x_i}(\vec{p})$, $i = 1, \dots, n$ (έχει αποδειχθεί)

$\not\Leftarrow$ π.χ. $f(x,y) = \begin{cases} 1 & xy = 0 \\ 0 & xy \neq 0 \end{cases}$ Ασυνεχής (δεν είναι διαφορ/μη)

$\exists\, \frac{\partial f}{\partial x}(0,0) = 0 = \frac{\partial f}{\partial y}(0,0)$




## (B) Εφαρμογή του :

$f : A \ (\subseteq \mathbb{R}^n) \to \mathbb{R}$, $\vec{p} \in A$ και

$\exists \ \nabla f(\vec{p}) = \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right)(\vec{p})$.

Τότε $\exists \ df(\vec{p}) \iff \lim\limits_{\vec{h} \to \vec{0}} \dfrac{f(\vec{p}+\vec{h}) - f(\vec{p}) - \nabla f(\vec{p}) \cdot \vec{h}}{\|\vec{h}\|} = 0$

(Σε διπλάδες συναρτήσεις)

1) $f(x,y) = \begin{cases} \dfrac{x^3}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Υπάρχει το διαφ της $f$ στο $(0,0)$;

$\frac{\partial f}{\partial x}(0,0) = \lim\limits_{h \to 0} \dfrac{f(0+h, 0) - f(0,0)}{h} =$

$= \lim\limits_{h \to 0} \dfrac{\frac{h^3}{h^2}}{h} = \lim\limits_{h \to 0} \dfrac{h^3}{h^3} = \lim\limits_{h \to 0} 1 = 1$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0, 0+h) - f(0,0)}{h} =$$

$$= \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$$

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$$\lim_{\vec{h}=(h_1,h_2)\to(0,0)} \frac{f(0+h_1, 0+h_2) - f(0,0) - \nabla f(0,0)\cdot(h_1,h_2)}{\|(h_1,h_2)\|} =$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{\frac{h^3}{h_1^2+h_2^2} - (h_1\cdot 1 + h_2\cdot 0)}{\sqrt{h_1^2+h_2^2}} =$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{-h_1 h_2^2}{(h_1^2+h_2^2)\sqrt{h_1^2+h_2^2}} \; (=0)$$

$h_1 = h_2 = h > 0$, $\displaystyle\lim_{h\to 0} \frac{-h^3}{h^2 h\sqrt{2}} = -\frac{1}{\sqrt{2}}$

Το όριο (δεν μας απασχολεί αν υπάρχει) **δεν** είναι 0

Άρα η f **δεν** είναι διαφορ. στο (0, 0)

2) $f(x,y) = \begin{cases} \frac{x^2 y}{\sqrt{x^2+y^2}} + 2x &, (x,y) \neq (0,0) \\ 0 &, (x,y) = (0,0) \end{cases}$

Ειναι διαφορ. στο $(0,0)$;

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$\frac{\partial f}{\partial x}(0,0) = \ldots = 2$

$\frac{\partial f}{\partial y}(0,0) = \ldots = 0$

$$\lim_{(h_1,h_2)\to(0,0)} \frac{f(0+h_1, 0+h_2) - f(0,0) - \nabla f(0,0)(h_1,h_2)}{\|(h_1,h_2)\|} =$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{\left(\frac{h_1^2 h_2}{\sqrt{h_1^2+h_2^2}} + 2h_1\right) - (2h_1 + 0h_2)}{\sqrt{h_1^2+h_2^2}} =$$

$$= \lim_{(h_1,h_2)\to(0,0)} \frac{h_1^2 h_2}{h_1^2+h_2^2} \quad *$$

$$\left|\frac{h_1^2 h_2}{h_1^2+h_2^2}\right| = \frac{|h_1|^2 |h_2|}{h_1^2+h_2^2} = \frac{|h_1|^2 |h_2|}{\|(h_1,h_2)\|^2} \leq \frac{\|(h_1,h_2)\|^2 \|(h_1,h_2)\|}{\|(h_1,h_2)\|^2} =$$

$= \|(h_1,h_2)\| \xrightarrow[(h_1,h_2)\to(0,0)]{} 0$ Άρα $* = 0$

Άρα $\exists\, df(0,0)(h_1,h_2) = 2h_1 + 0\cdot h_2$

$\overset{*}{3)}$ Έστω $f(x,y) = \begin{cases} (x^2+y^2)\,\text{ημ}\dfrac{1}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

i) $\exists \dfrac{\partial f}{\partial x}(x_0, y_0), \dfrac{\partial f}{\partial y}(x_0, y_0) \ \forall (x_0, y_0) \in \mathbb{R}^2$

ii) Να αποδείξετε ότι αυτές είναι ασυνεχείς στο $(0,0)$

iii) $\exists\, df(0,0)(h_1, h_2) = 0h_1 + 0h_2$.

• Σημείωση: Από αυτή την άσκ. και από τις προηγ. έχουμε:


Λύση i) $\dfrac{\partial f}{\partial x}(x,y) = \begin{cases} 2x\,\text{ημ}\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) - \dfrac{x}{\sqrt{x^2+y^2}}\,\text{συν}\dfrac{1}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ \lim\limits_{h \to 0} \dfrac{f(0+h,0)-f(0,0)}{h} = \lim\limits_{h \to 0} h\,\text{ημ}\dfrac{1}{|h|} = 0, & (x,y) = (0,0) \end{cases}$

ii) Ασυνεχής στο $(0,0)$: Διαδρομή $(x,0)$ με $x>0$, $\lim\limits_{x \to 0^+} \dfrac{\partial f}{\partial x}(x,0) = 0 - \underbrace{\lim\limits_{x \to 0^+} \text{συν}\dfrac{1}{x}}_{\text{δεν υπάρχει}}$

Ανάλογα η $\dfrac{\partial f}{\partial y}(x,y)$

iii) $\lim\limits_{(h_1,h_2) \to (0,0)} \dfrac{f(0+h_1, 0+h_2) - f(0,0) - \nabla f(0,0) \cdot (h_1, h_2)}{\|(h_1,h_2)\|} =$

$= \lim\limits_{(h_1,h_2) \to (0,0)} \sqrt{h_1^2+h_2^2}\,\underbrace{\text{ημ}\dfrac{1}{\sqrt{h_1^2+h_2^2}}}_{\text{(φραγμένη)}} = 0$. Άρα $\exists\, df(0,0)(h_1,h_2) = 0 \cdot h_1 + 0 \cdot h_2$

# Παράγωγοι Ανωτέρας Τάξης

$f : A(\subseteq \mathbb{R}^n) \to \mathbb{R}$

Έστω $\exists\, g_i = \dfrac{\partial f}{\partial x_i} : A(\subseteq \mathbb{R}^n) \to \mathbb{R}$

Εάν $\exists\, \dfrac{\partial g_i}{\partial x_j}(\vec{p}) \overset{\text{συμ.}}{=} \dfrac{\partial^2 f}{\partial x_j \partial x_i}(\vec{p})$

Αν $i=j$, $\dfrac{\partial^2 f}{\partial x_i \partial x_i} \overset{\text{συμ.}}{=} \dfrac{\partial^2 f}{\partial x_i^2}$

Ανάλογα $\dfrac{\partial^3 f}{\partial x_k \partial x_j \partial x_i}$ κ.ο.κ.

## Άσκηση

$f(x,y) = x + y^2 + e^x συνy$. Οι $\dfrac{\partial^2 f}{\partial x^2}$, $\dfrac{\partial^2 f}{\partial y^2}$, $\dfrac{\partial^2 f}{\partial x \partial y}$, $\dfrac{\partial^2 f}{\partial y \partial x}$ να υπολογισθούν.

Τι συμβαίνει;

$\dfrac{\partial f}{\partial x}(x,y) = 1 + e^x συνy$, $\dfrac{\partial^2 f}{\partial x^2}(x,y) =$

$= e^x συνy$, $\dfrac{\partial^2 f(x,y)}{\partial y \partial x} = -e^x ημy$

$\dfrac{\partial f}{\partial y}(x,y) = 2y - e^x ημx$, $\dfrac{\partial^2 f}{\partial y^2}(x,y) = 2 - e^x συνy$,

$$\frac{\partial^2 f}{\partial x \partial y}(x,y) = -e^x \eta\mu y \, ,$$

Παρατηρούμε ότι $\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial^2 f}{\partial y \partial x}(x,y)$

Δεν συμβαίνει πάντα!!!

<u>Θεώρημα Clairaut ή Μεικτών Παραγώγων</u>

Εάν $\exists$ $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y \partial x}$ και είναι συνεχείς

τότε είναι ίσες!

<u>Παράδειγμα</u>: όπου $\frac{\partial^2 f}{\partial x \partial y}(x,y) \neq \frac{\partial^2 f}{\partial y \partial x}(x,y)$

$$f(x,y) = \begin{cases} xy\,\dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

<u>ΛΥΣΗ</u>

$$\frac{\partial f}{\partial x}(0,y) = \begin{cases} -\frac{y^5}{y^4} = -y & , y \neq 0 \\ 0 & , y = 0 \end{cases} = -y \, , \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(0,0) = -1$$

$$\frac{\partial f}{\partial y}(x,0) = x \, , \quad \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(0,0) = 1$$